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Bounded and periodic solutions of infinite delay evolution equations

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Abstract

For $A(t)$ and $f(t,x,y)$ T-periodic in t, we consider the following evolution equation with infinite delay in a general Banach space X:

$$u'(t) + A(t)u(t) = f\big(t, u(t), u_t\big), \quad t > 0,\ u(s) = \phi(s),\ s \leqslant 0, \tag{0.1}$$

where the resolvent of the unbounded operator $A(t)$ is compact, and $u_t(s) = u(t+s)$, $s \leqslant 0$. By utilizing a recent asymptotic fixed point theorem of Hale and Lunel (1993) for condensing operators to a phase space C_g, we prove that if solutions of Eq. (0.1) are ultimate bounded, then Eq. (0.1) has a T-periodic solution. This extends and improves the study of deriving periodic solutions from boundedness and ultimate boundedness of solutions to infinite delay evolution equations in general Banach spaces; it also improves a corresponding result in J. Math. Anal. Appl. 247 (2000) 627–644 where the local strict boundedness is used.

Keywords: Infinite delay; Bounded and periodic solutions; Condensing operators; Hale and Lunel's fixed point theorem

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1. Introduction

This paper is concerned with deriving periodic solutions from ultimate boundedness of solutions for the following infinite delay evolution equation:

$$u'(t) + A(t)u(t) = f\big(t, u(t), u_t\big), \quad t > 0,\ u(s) = \phi(s),\ s \leqslant 0, \tag{1.1}$$

in a general Banach space $(X, \|\cdot\|)$, where $A(t)$ is a unbounded operator, and $A(t)$ and $f(t, x, y)$ are T-periodic in t. Here $u_t \in C((-\infty, 0], X)$ (space of continuous functions on $(-\infty, 0]$ with values in X) is defined by $u_t(s) = u(t+s)$, $s \leqslant 0$.

A standard approach in deriving T-periodic solutions is to define the Poincare operator [1] given by

$$P(\phi) = u_T(\phi),$$

which maps an initial function (or value) ϕ along the unique solution $u(\phi)$ by T-units. Then conditions are given such that some fixed point theorem can be applied to get fixed points for the Poincare operator, which give rise to periodic solutions.

In [7], a phase space C_g is constructed in order to study Eq. (1.1); and it is proved that in C_g the Poincare operator for Eq. (1.1) is a condensing operator with respect to Kuratowski's measure of non-compactness. Therefore, Sadovskii's (or Darbo's [3]) fixed point theorem is used to get fixed points of the operator P and hence T-periodic solutions of Eq. (1.1). In using Sadovskii's fixed point theorem, it is required that the Poincare operator maps some set into itself. Therefore, a notion called "local strict boundedness" (see Definition 4.1 in [7]) is introduced to fulfill this requirement. Local strict boundedness basically says that solutions started initially from a set will remain in the same set, thus it requires more than the conditions of boundedness and ultimate boundedness.

Recently, after analyzing Eq. (1.1) with the same structure as in [7] (so that the Poincare operator is condensing), we find that the techniques used in [5–7] and a recent asymptotic fixed point theorem due to Hale and Lunel [3] for condensing operators (which is an extension of Browder's asymptotic fixed point theorem for completely continuous operators) can be employed to obtain a direct extension of the classical results in this area. That is, we are able to prove that if solutions of Eq. (1.1) are bounded and ultimate bounded, then the Poincare operator has a fixed point and hence Eq. (1.1) has a T-periodic solution. This way, the earlier studies of deriving periodic solutions from boundedness and ultimate boundedness for evolution equations without delay or with finite delay can be carried to evolution equations with infinite delay in general Banach spaces. It also improves a corresponding result in [7] where the local strict boundedness is used.

After this, we will study the relationship between boundedness and ultimate boundedness. We first reduce the requirement of boundedness by introducing a notion called "local boundedness" (see Definition 3.1), and show that {local boundedness and ultimate boundedness} is equivalent to {boundedness and ultimate boundedness}. Finally, we show that for Eq. (1.1) (with the same structure as in [7]) and some other equations, local boundedness holds. So that for Eq. (1.1) and some other equations, ultimate boundedness alone implies boundedness and ultimate boundedness, which in turn implies the existence of periodic solutions (see Theorem 3.4). This improves and simplifies many earlier results for which boundedness and ultimate boundedness are assumed in order to obtain periodic solutions.

We will study periodic solutions in Section 2, and study the relationship between boundedness and ultimate boundedness in Section 3.

2. Periodic solutions

In this section we study periodic solutions for Eq. (1.1). We make the following assumptions.

Assumption 2.1. For a constant $T > 0$, $f(t+T, x, y) = f(t, x, y)$, $A(t+T) = A(t)$, $t \geqslant 0$. f is continuous in its variables and is locally Lipschitzian in the second and the third variables, and f maps bounded sets into bounded sets.

Assumption 2.2 [8, p. 150]. For $t \in [0, T]$ one has

(H1) The domain $D(A(t)) = D$ is independent of t and is dense in X.
(H2) For $t \geqslant 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $\operatorname{Re}\lambda \leqslant 0$ and is compact, and there is a constant M independent of λ and t such that

$$\big\|R\big(\lambda, A(t)\big)\big\| \leqslant M\big(|\lambda| + 1\big)^{-1}, \quad \operatorname{Re}\lambda \leqslant 0.$$

(H3) There exist constants $L > 0$ and $0 < a \leqslant 1$ such that

$$\big\|\big(A(t) - A(s)\big)A(r)^{-1}\big\| \leqslant L|t-s|^a, \quad s, t, r \in [0, T].$$

Under these assumptions, the results in, e.g., [1,8] imply the existence of a unique evolution system $U(t, s)$, $0 \leqslant s \leqslant t \leqslant T$, for Eq. (1.1).

Now, we define the phase space C_g for Eq. (1.1). First, we have, from [7],

Lemma 2.1 [7, Lemma 2.1]. *There exists an integer $K_0 > 1$ such that*

$$\left(\frac{1}{2}\right)^{K_0-1} M_0 < 1, \tag{2.1}$$

where $M_0 = \sup_{t\in[0,T]} \|U(t,0)\|$ is finite. Next, let $w_0 = T/K_0$; then there exists a function g on $(-\infty, 0]$ such that $g(0) = 1$, $g(-\infty) = \infty$, g is decreasing on $(-\infty, 0]$, and for $d \geqslant w_0$ one has

$$\sup_{s\leqslant 0} \frac{g(s)}{g(s-d)} \leqslant \frac{1}{2}. \tag{2.2}$$

Based on the above function g, the space

$$C_g = \left\{\phi\colon\ \phi \in C\big((-\infty, 0], X\big) \text{ and } \sup_{s\leqslant 0} \frac{\|\phi(s)\|}{g(s)} < \infty\right\} \tag{2.3}$$

is well defined and is a Banach space with the norm

$$|\phi|_g = \sup_{s \leqslant 0} \frac{\|\phi(s)\|}{g(s)}, \quad \phi \in C_g. \tag{2.4}$$

Concerning the solutions of Eq. (1.1), we have, from [7],

Theorem 2.1 [7, Theorem 2.1]. *Let Assumptions* 2.1 *and* 2.2 *be satisfied, and let* $\phi \in C_g$ *be fixed. Then there exists a constant* $\alpha > 0$ *and a unique continuous function* $u : (-\infty, \alpha] \to X$ *such that* $u_0 = \phi$ *(i.e.,* $u(s) = \phi(s)$, $s \leqslant 0$*), and*

$$u(t) = U(t,0)\phi(0) + \int_0^t U(t,h) f\big(h, u(h), u_h\big)\, dh, \quad t \in [0, \alpha]. \tag{2.5}$$

A function satisfying (2.5) is called a mild solution of Eq. (1.1). Thus Theorem 2.1 says that mild solutions exist and are unique for Eq. (1.1). In the sequel, we follow [4,7] and other related papers and call "mild solutions" as "solutions." We also assume that solutions exist on $[0, \infty)$ in order to study periodic solutions; and we use $u(\cdot, \phi)$ to denote the unique solution with the initial function ϕ.

Now, consider the Poincare operator $P : C_g \to C_g$ given by

$$P(\phi) = u_T(\cdot, \phi), \quad \phi \in C_g, \tag{2.6}$$

i.e., $(P\phi)(s) = u_T(s, \phi) = u(T + s, \phi)$, $s \leqslant 0$, which maps the initial function ϕ along the unique solution $u(\cdot, \phi)$ by T units.

Definition 2.1 [3]. Suppose that α is Kuratowski's measure of non-compactness in Banach space Y and that $P : Y \to Y$ is a continuous operator. Then P is said to be a *condensing operator* if P takes bounded sets into bounded sets, and $\alpha(P(B)) < \alpha(B)$ for every bounded set B of Y with $\alpha(B) > 0$.

The following result is proved in [7].

Theorem 2.2 [7, Theorem 4.1]. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Then the operator* P *defined by* (2.6) *is condensing in* C_g *with* g *given in Lemma* 2.1.

Next, we state a recent asymptotic fixed point theorem due to Hale and Lunel [3] for condensing operators, which is an extension of Browder's asymptotic fixed point theorem for completely continuous operators.

Theorem 2.3 [3]. *Suppose* $S_0 \subseteq S_1 \subseteq S_2$ *are convex bounded subsets of a Banach space* Y, S_0 *and* S_2 *are closed, and* S_1 *is open in* S_2, *and suppose* $P : S_2 \to Y$ *is* (S_2) *condensing in the following sense: if* U *and* $P(U)$ *are contained in* S_2 *and* $\alpha(U) > 0$, *then* $\alpha(P(U)) < \alpha(U)$. *If* $P^j(S_1) \subseteq S_2$, $j \geqslant 0$, *and, for any compact set* $H \subseteq S_1$, *there is a number* $N(H)$ *such that* $P^k(H) \subseteq S_0$, $k \geqslant N(H)$, *then* P *has a fixed point.*

Based on this, we deduce the following asymptotic fixed point theorem for condensing operators.

Theorem 2.4. *Suppose $S_0 \subseteq S_1 \subseteq S_2$ are convex bounded subsets of a Banach space Y, S_0 and S_2 are closed, and S_1 is open in S_2, and suppose P is a condensing operator in Y. If $P^j(S_1) \subseteq S_2$, $j \geqslant 0$, and there is a number $N(S_1)$ such that $P^k(S_1) \subseteq S_0$, $k \geqslant N(S_1)$, then P has a fixed point.*

Notice that the statement in Theorem 2.4 is similar to that of Browder's or Horn's asymptotic fixed point theorem. But the difference is that Theorem 2.4 does not involve compactness, and therefore is particularly useful here because, as discussed in [7], under the Poincare operator P with infinite delay, an initial function on $(-\infty, 0]$ becomes a segment on $(-\infty, 0]$ of a function defined on $(-\infty, T]$. Thus compactness is not applicable now to the Poincare operator P, hence Browder's or Horn's asymptotic fixed point theorem (which involves compactness) cannot be used here to deal with infinite delay.

Next, we state the definitions of boundedness and ultimate boundedness [2] and show, by using Theorem 2.4, that they can be used to derive the existence of periodic solutions.

Definition 2.2. The solutions of Eq. (1.1) are said to be *bounded* if for each $B_1 > 0$, there is $B_2 > 0$, such that $|\phi|_g \leqslant B_1$ and $t \geqslant 0$ imply that its solution satisfies $\|u(t, \phi)\| < B_2$.

Definition 2.3. The solutions of Eq. (1.1) are said to be *ultimate bounded* if there is a bound $B > 0$, such that for each $B_3 > 0$, there is $K > 0$, such that $|\phi|_g \leqslant B_3$ and $t \geqslant K$ imply that its solution satisfies $\|u(t, \phi)\| < B$.

Theorem 2.5. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. If the solutions of Eq.* (1.1) *are bounded and ultimate bounded, then Eq.* (1.1) *has a T-periodic solution.*

Proof. Let the operator P be defined by (2.6). From [7], we have

$$P^m(\phi) = u_{mT}(\phi), \quad \phi \in C_g,\ m = 0, 1, 2, \ldots. \tag{2.7}$$

Next, let $B > 0$ be the bound in the definition of ultimate boundedness. Using boundedness, there is $B_1 > B$ such that $\{|\phi|_g \leqslant B,\ t \geqslant 0\}$ implies $\|u(t, \phi)\| < B_1$. Also, there is $B_2 > B_1$ such that $\{|\phi|_g \leqslant B_1,\ t \geqslant 0\}$ implies $\|u(t, \phi)\| < B_2$. Next, using ultimate boundedness, there is a positive integer J such that $\{|\phi|_g \leqslant B_1,\ t \geqslant JT\}$ implies $\|u(t, \phi)\| < B$.

Now let

$$\begin{aligned}
&S_2 \equiv \bigl\{\phi \in C_g\colon\ |\phi|_g \leqslant B_2\bigr\},\\
&W \equiv \bigl\{\phi \in C_g\colon\ |\phi|_g < B_1\bigr\}, \qquad S_1 \equiv W \cap S_2,\\
&S_0 \equiv \bigl\{\phi \in C_g\colon\ |\phi|_g \leqslant B\bigr\},
\end{aligned} \tag{2.8}$$

so that $S_0 \subseteq S_1 \subseteq S_2$ are convex bounded subsets of Banach space C_g, S_0 and S_2 are closed, and S_1 is open in S_2. Next, for $\phi \in S_1$ and $j \geqslant 0$,

$$\begin{aligned}
|P^j\phi|_g &= \bigl|u_{jT}(\phi)\bigr|_g = \sup_{s \leqslant 0} \frac{\|u_{jT}(s)\|}{g(s)} = \sup_{s \leqslant 0} \frac{\|u(jT+s)\|}{g(s)}\\
&\leqslant \max\left\{\sup_{s \leqslant -jT} \frac{\|u(jT+s)\|}{g(s)},\ \sup_{s \in [-jT, 0]} \frac{\|u(jT+s)\|}{g(s)}\right\}
\end{aligned}$$

$$\leqslant \max\left\{\sup_{l\leqslant 0}\frac{\|u(l)\|}{g(l-jT)},\ \sup_{l\in[0,jT]}\big\|u(l)\big\|\right\}$$
$$\leqslant \max\left\{\sup_{l\leqslant 0}\frac{\|u(l)\|}{g(l)},\ \sup_{l\in[0,jT]}\big\|u(l)\big\|\right\} \leqslant \max\big\{|\phi|_g, B_2\big\} \leqslant B_2, \tag{2.9}$$

which implies $P^j(S_1) \subseteq S_2$, $j \geqslant 0$. Now, we prove that there is a number $N(S_1)$ such that $P^k(S_1) \subseteq S_0$ for $k \geqslant N(S_1)$. To this end, we choose a positive integer $m = m(B_1)$ such that

$$\left(\frac{1}{2}\right)^m < \frac{B}{B_1}, \tag{2.10}$$

and then choose an integer $N = N(S_1) > J$ such that

$$NT > mw_0 \quad \text{and} \quad \frac{B_2}{g(-(N-J)T)} < B, \tag{2.11}$$

where w_0 is from Lemma 2.1. Then for $\phi \in S_1$ and $k \geqslant N$,

$$|P^k\phi|_g = \big|u_{kT}(\phi)\big|_g = \sup_{s\leqslant 0}\frac{\|u_{kT}(s)\|}{g(s)} = \sup_{s\leqslant 0}\frac{\|u(kT+s)\|}{g(s)}$$
$$\leqslant \max\left\{\sup_{s\leqslant -kT}\frac{\|u(kT+s)\|}{g(s)},\ \sup_{s\in[-kT,-(k-J)T]}\frac{\|u(kT+s)\|}{g(s)},\right.$$
$$\left.\sup_{s\in[-(k-J)T,0]}\frac{\|u(kT+s)\|}{g(s)}\right\}. \tag{2.12}$$

For the terms in (2.12), we have

$$\sup_{s\in[-(k-J)T,0]}\frac{\|u(kT+s)\|}{g(s)} \leqslant \sup_{l\in[JT,kT]}\big\|u(l)\big\| < B, \tag{2.13}$$

and

$$\sup_{s\in[-kT,-(k-J)T]}\frac{\|u(kT+s)\|}{g(s)} \leqslant \sup_{l\in[0,JT]}\frac{\|u(l)\|}{g(l-kT)}$$
$$\leqslant \frac{B_2}{g(-(k-J)T)} \leqslant \frac{B_2}{g(-(N-J)T)} < B, \tag{2.14}$$

and

$$\sup_{s\leqslant -kT}\frac{\|u(kT+s)\|}{g(s)} = \sup_{l\leqslant 0}\frac{\|u(l)\|}{g(l-kT)}$$
$$= \sup_{l\leqslant 0}\frac{\|u(l)\|}{g(l)}\frac{g(l)}{g(l-kT)} \leqslant |\phi|_g \sup_{l\leqslant 0}\frac{g(l)}{g(l-kT)}$$
$$\leqslant B_1 \sup_{l\leqslant 0}\frac{g(l)}{g(l-w_0)}\frac{g(l-w_0)}{g(l-2w_0)}\cdots\frac{g(l-(m-1)w_0)}{g(l-mw_0)}\frac{g(l-mw_0)}{g(l-kT)}. \tag{2.15}$$

Now, from Lemma 2.1, for $i \geqslant 0$,

$$\sup_{l\leqslant 0}\frac{g(l-iw_0)}{g(l-(i+1)w_0)}=\sup_{s\leqslant -iw_0}\frac{g(s)}{g(s-w_0)}\leqslant \sup_{s\leqslant 0}\frac{g(s)}{g(s-w_0)}\leqslant \frac{1}{2}. \tag{2.16}$$

Thus, (2.15) becomes

$$\begin{aligned}\sup_{s\leqslant -kT}\frac{\|u(kT+s)\|}{g(s)}&\leqslant B_1\left(\frac{1}{2}\right)^m\sup_{l\leqslant 0}\frac{g(l-mw_0)}{g(l-kT)}\\&< B_1\frac{B}{B_1}\sup_{l\leqslant 0}\frac{g(l-mw_0)}{g(l-NT)}\leqslant B\sup_{l\leqslant 0}\frac{g(l-mw_0)}{g(l-mw_0)}=B.\end{aligned} \tag{2.17}$$

Therefore, (2.12) becomes

$$|P^k\phi|_g\leqslant B,\quad k\geqslant N, \tag{2.18}$$

which implies $P^k(S_1)\subseteq S_0$, $k\geqslant N(S_1)$. Now, Theorem 2.4 can be used to obtain a fixed point for the operator P, which, from [7], gives rise to a T-periodic solution of Eq. (1.1). This proves the theorem. □

3. Boundedness and ultimate boundedness

In this section, we will study the relationship between boundedness and ultimate boundedness. To this end, we introduce the following notion of "local boundedness," which will reduce the requirement of the boundedness.

Definition 3.1. The solutions of Eq. (1.1) are said to be *locally bounded* if for each $B_1>0$ and $K>0$, there is $B_2>0$, such that $|\phi|_g\leqslant B_1$ and $0\leqslant t\leqslant K$ imply that its solution satisfies $\|u(t,\phi)\|<B_2$.

Theorem 3.1. {*Local boundedness and ultimate boundedness*} *implies* {*boundedness and ultimate boundedness*}.

Proof. We only need to prove the boundedness. Let $B>0$ be the bound in the definition of ultimate boundedness. For any $B_1>0$, from the ultimate boundedness, there is $K>0$ such that $|\phi|_g\leqslant B_1$ and $t\geqslant K$ imply $\|u(t,\phi)\|<B$. Next, solutions are locally bounded, so that for the given $B_1>0$ and $K>0$, there is $B_2>B$ such that $|\phi|_g\leqslant B_1$ and $0\leqslant t\leqslant K$ imply $\|u(t,\phi)\|<B_2$. Now, it is clear that $|\phi|_g\leqslant B_1$ and $t\geqslant 0$ imply $\|u(t,\phi)\|<B_2$, which proves the boundedness. □

Accordingly, we can restate Theorem 2.5 as follows.

Theorem 3.2. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. If the solutions of Eq.* (1.1) *are locally bounded and ultimate bounded, then Eq.* (1.1) *has a T-periodic solution.*

Next, note that with some conditions on the function f, such as Lipschitzian conditions, it is shown in [7] that the solutions of Eq. (1.1) are indeed locally bounded.

Theorem 3.3 [7, Theorem 2.2]. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. Then the solutions of Eq.* (1.1) *are locally bounded.*

Therefore, using Theorems 3.2 and 3.3, we conclude that for Eq. (1.1), ultimate boundedness alone implies the existence of T-periodic solutions, which is stated below.

Theorem 3.4. *Let Assumptions* 2.1 *and* 2.2 *be satisfied. If the solutions of Eq.* (1.1) *are ultimate bounded, then Eq.* (1.1) *has a T-periodic solution.*

Note that in [7], the local boundedness is proven using the Lipschitzian conditions and Gronwall's inequality on finite intervals. Therefore, the local boundedness will hold for a large class of differential equations and integrodifferential equations if similar conditions are assumed. Consequently, for those equations, ultimate boundedness alone implies the existence of periodic solutions. This result improves and simplifies many earlier results for which boundedness and ultimate boundedness are assumed in order to derive periodic solutions.

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