

On the Bounded Solutions of Volterra Equations*

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We extend the method of sums of commuting operators to the study of the existence and uniqueness of bounded solutions of Volterra equations of the form $\dot{u}(t) = Au(t) + \int_0^\infty dB(\tau)u(t-\tau) + f(t)$ with bounded f in the infinite dimensional case. The main results are necessary and sufficient conditions for the above equations to have a unique bounded solution with spectrum not intersecting the spectrum of the equation under consideration. Applications are made to illustrate the main results.

Keywords: Volterra equations; Sums of commuting operators; Spectrum of functions; Bounded solutions; Periodic solutions; Almost periodic solutions

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1. INTRODUCTION

In this article we are concerned with Volterra equations of the form

$$\dot{u}(t) = Au(t) + \int_0^\infty dB(\tau)u(t-\tau) + f(t),$$
(1)

where *f* is a bounded continuous (or almost periodic) function, *A* is in general *unbounded closed linear operator* on a Banach space X, with the domain D(A) and (*the standing condition*) $\{B(t)\}_{t\geq 0}$ is a family of closed linear operators in X, $B(\cdot) : \mathbb{R}_+ \to L(\mathbb{Y}, \mathbb{X})$ is left continuous and of bounded variation, where \mathbb{Y} is D(A) with the graph norm $||y|| := ||Ay|| + ||y||, \forall y \in D(A)$.

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The study of bounded and almost periodic solutions of linear inhomogeneous equations is one of central topics of the qualitative theory of differential equations which plays an important role in investigating the behavior of solutions of the perturbed non-linear equations (see e.g. [6,9,10,14,15,26]). As is well known, many problems of applied mathematics lead to equations of the form (1) (see for instance [4,7,10,14,20]). The global theory of (1) is concerned with the existence of periodic, almost periodic solutions of (1) with f of similar property.

Recently, of increasing interest is the extension of the classical results in the finite dimensional case to the infinite dimensional case (see [1,2,5,8,13,16,17,19-22,23,24] and the references therein for more information). The general setting of the problem stated above has been first studied by Prüss [19]. Necessary and sufficient conditions for several classes of equations, including the case where the spectrum of *f* is compact, have been obtained. For example, when *A* is the generator of an analytic semigroup the main task in [19] is to try to prove the following condition

$$\Lambda \cap \Lambda_0 = \emptyset, \tag{2}$$

where $\Lambda_0 := \{\xi \in \mathbb{R}: \exists (i\xi - A - dB(\xi))^{-1} \in L(\mathbb{X}, \mathbb{Y})\}$ (which is called *spectrum* of Eq. (1), here $dB(\xi)$ stands for the Fourier transform of dB), is a necessary and sufficient condition for the admissibility of the function space $\Lambda(\mathbb{X})$ for Eq. (1). However, as pointed out in [19, p. 136], the problem of finding necessary and sufficient spectral conditions for the existence of bounded solutions to the general class of equations with *B* of bounded variation is open. For example, the question of whether (2) is necessary and sufficient for the existence and uniqueness of periodic solution to the following simplest equation

$$\dot{u}(t) = Au(t) + Bu(t-1) + f(t), \quad t \in \mathbb{R},$$
(3)

where A is the generator of an analytic semigroup and B is an arbitrary bounded linear operator, is still open. Needless to say, this kind of problem has many applications in physics, biology, etc. In [8,21] necessary and sufficient conditions for the existence of periodic solutions to Eq. (1) have been given with some additional conditions on the phase spaces or f. We refer the reader to [11,12] and the references therein for complete results in the finite dimensional case.

In this article we will extend the abstract approach of sums of commuting operators as done in [16] in combination with the method of Prüss in [19] to give a necessary and sufficient condition for the existence of bounded solutions to Eq. (1). Our results can be applied to solve completely the above kind of problems (3) with *B* even more general. Roughly speaking, our method of study is first to decompose *f* into two components one of which has a compact spectrum and the other one has a spectrum far enough from zero. Then we solve the equation with the first component by using Prüss' method. The second component can be dealt with by using the sums of the commuting operators method. Finally, the superposition principle allows us to get the bounded solution for the starting equation by summing up the two solutions obtained above. The main result obtained in this article is Theorem 4. Its particular case, Theorem 5 resolves completely the bounded solution problem for the case of bounded B(t). Applications are provided in Section 5. The results of this article complement and extend several results of [11,12,15,16,19,25].

2. PRELIMINARIES

2.1. Notation

Throughout the article we will use the following notations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ stand for the set of natural, integer, real, complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. X will denote a given complex Banach space. If *T* is a linear operator on X, then D(T)stands for its domain. Given two Banach spaces \mathbb{Y}, \mathbb{W} by $L(\mathbb{Y}, \mathbb{W})$ we will denote the space of all bounded linear operators from \mathbb{Y} to \mathbb{W} and $L(\mathbb{X}, \mathbb{X}) := L(\mathbb{X})$. As usual, $\sigma(T), \rho(T), R(\lambda, T)$ are the notations of the spectrum, resolvent set, and resolvent of the operator *T*, respectively. We will denote by $\sigma_i(T) := \{\xi \in \mathbb{R}: i\xi \in \sigma(T)\}$. The notations BUC(\mathbb{R}, \mathbb{X}), AP(\mathbb{X}) will stand for the space of all \mathbb{X} -valued bounded uniformly continuous functions on \mathbb{R} and its subspace of almost periodic functions in Bohr's sense. By $(S(t))_{t\geq 0}$ we will denote the translation group on BUC(\mathbb{R}, \mathbb{X}), i.e., $S(t)v(s) := v(t+s), \forall t, s \in \mathbb{R}, v \in BUC(\mathbb{R}, \mathbb{X})$ with infinitesimal generator $\mathcal{D} := d/dt$ which is defined on $D(\mathcal{D}) := BUC^1(\mathbb{R}, \mathbb{X})$. Let \mathcal{M} be a subspace of BUC(\mathbb{R}, \mathbb{X}), Abe a linear operator on \mathbb{X} . We shall denote by $\mathcal{A}_{\mathcal{M}}$ the operator $f \in \mathcal{M} \mapsto Af(\cdot)$ with $D(\mathcal{A}_{\mathcal{M}}) = \{f \in \mathcal{M} | \forall t \in \mathbb{R}, f(t) \in D(A), Af(\cdot) \in \mathcal{M}\}$. When $\mathcal{M} = \Lambda(\mathbb{X})$ (see the definition of this function space in the next subsection) we shall use the notation $\mathcal{A} := \mathcal{A}_{\mathcal{M}}$.

2.2. Spectrum of a Bounded Uniformly Continuous Function

The spectrum of a given function $u \in BUC(\mathbb{R}, \mathbb{X})$ is defined as the following set *(Beurling spectrum)*

$$sp(u) := \{\xi \in \mathbb{R} : \forall \epsilon > 0 \ \exists f \in L^1(\mathbb{R}), supp \ \mathcal{F}f \subset (\xi - \epsilon, \xi + \epsilon), \ f * u \neq 0\},$$
(4)

where

$$f * u(s) := \int_{-\infty}^{+\infty} f(s-t)u(t) dt \qquad \mathcal{F}f(s) := \int_{-\infty}^{\infty} e^{-ist}f(t) dt.$$

It coincides with the set (*Carleman spectrum*) consisting of $\xi \in \mathbb{R}$ such that the Fourier–Carleman transform of u

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t) \, dt, & (\operatorname{Re} \lambda > 0) ; \\ -\int_0^\infty e^{\lambda t} u(-t) \, dt, & (\operatorname{Re} \lambda < 0) \end{cases}$$
(5)

has no holomorphic extension to a neighborhood of $i\xi$ (see e.g. [20, Proposition 0.5, p. 22]). In turn, the Carleman spectrum of a uniformly continuous and bounded function u coincides with its *Arveson spectrum*

$$isp(u) = \sigma(\mathcal{D}_{\mathcal{M}_u}),$$
 (6)

where \mathcal{M}_u is the closed subspace of BUC(\mathbb{R}, \mathbb{X}) spanned by all translations of u, i.e.,

$$\mathcal{M}_u := \overline{\operatorname{span}\{S(\tau)u, \tau \in \mathbb{R}\}},\tag{7}$$

(see [2, Section 2] for a short introduction to these notions of spectrum and its inter-relations).

We collect some main properties of the spectrum of a function, which we will need in the sequel, for the reader's convenience.

THEOREM 1 Let $f, g_n \in BUC(\mathbb{R}, \mathbb{X}), n \in \mathbb{N}$, such that $g_n \to f$ as $n \to \infty, \psi \in S$, where S is the Schwartz space of all C^{∞} -functions on \mathbb{R} with each of its derivatives decaying faster than any polynomial. Then

- (i) sp(f) is closed,
- (ii) $\operatorname{sp}(f(\cdot + h)) = \operatorname{sp}(f)$,
- (iii) If $\alpha \in \mathbb{C} \setminus \{0\}$ sp $(\alpha f) = \operatorname{sp}(f)$,
- (iv) If $\operatorname{sp}(g_n) \subset \Lambda$ for all $n \in \mathbb{N}$ then $\operatorname{sp}(f) \subset \overline{\Lambda}$,
- (v) If A is a closed operator, $f(t) \in D(A)$, $\forall t \in \mathbb{R}$ and $Af \in BUC(\mathbb{R}, \mathbb{X})$, then $sp(Af) \subset sp(f)$,
- (vi) $\operatorname{sp}(\psi * f) \subset \operatorname{sp}(f) \cap \operatorname{supp}\mathcal{F}\psi$,
- (vii) $\operatorname{sp}(f \psi * f) \subset \operatorname{sp}(f) \cap \operatorname{supp}(1 \mathcal{F}\psi),$
- (viii) For $K \in BV(L(\mathbb{Y}, \mathbb{X}))$, $sp(dK * f) \subset sp(f)$.

Proof For the proof we refer the reader to [20, Proposition 0.4, p. 20, Proposition 0.6, p. 25, Theorem 0.8, p. 21] and [19, Proposition 2]. ■

Definition 1 A closed and translation invariant subspace \mathcal{M} of the function space BUC(\mathbb{R}, \mathbb{X}), i.e., $S(\tau)\mathcal{M} \subset \mathcal{M}$ for all $\tau \in \mathbb{R}$, is said to satisfy *condition H* if the following condition is fulfilled:

$$\forall C \in L(\mathbb{X}), \quad \forall f \in \mathcal{M} \Rightarrow \mathcal{C}f \in \mathcal{M}.$$

In the article, as a model of the translation invariant subspaces, which satisfy condition H, we can take

$$\Lambda(\mathbb{X}) := \{ u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X}) \colon \mathrm{sp}(u) \subset \Lambda \},\tag{8}$$

where Λ is a given closed subset of the real line. In connection with the translation invariant subspaces we need the following simple spectral properties.

LEMMA 1 Let \mathcal{M} satisfy condition H. Then $\sigma(\mathcal{A}_{\mathcal{M}}) \subset \sigma(A)$ and

$$\|R(\lambda, \mathcal{A}_{\mathcal{M}})\| \le \|R(\lambda, A)\|, \quad \forall \lambda \in \rho(A).$$

Proof Let $\lambda \in \rho(A)$. We show that $\lambda \in \rho(A_M)$. In fact, as \mathcal{M} satisfies condition H, $\forall f \in \mathcal{M}, R(\lambda, A)f(\cdot) := (\lambda - A)^{-1}f(\cdot) \in \mathcal{M}$. Thus the function $R(\lambda, A)f(\cdot)$ is a solution to the equation $(\lambda - A_M)u = f$. Moreover, since $\lambda \in \rho(A)$ it is seen that the above equation has at most one solution. Hence $\lambda \in \rho(A_M)$. Obviously, $||R(\lambda, A_M)|| \le ||R(\lambda, A)||$.

2.3. Sums of Commuting Operators

We recall now the notion of two commuting operators which will be used in the sequel.

Definition 2 Let A and B be operators on a Banach space G with nonempty resolvent set. We say that A and B commute if one of the following equivalent conditions hold:

- (i) $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for some (all) $\lambda \in \rho(A), \mu \in \rho(B)$,
- (ii) $x \in D(A)$ implies $R(\mu, B)x \in D(A)$ and $AR(\mu, B)x = R(\mu, B)Ax$ for some (all) $\mu \in \rho(B)$.

For
$$\theta \in (0, \pi)$$
, $R > 0$ we denote $\Sigma(\theta, R) = \{z \in \mathbb{C} : |z| \ge R, |\arg z| \le \theta\}$.

Definition 3 Let A and B be commuting operators. Then

(i) A is said to be of class $\Sigma(\theta, R)$ if there are positive constant R and

$$\Sigma(\theta, R) \subset \rho(A)$$
 and $\sup_{\lambda \in \Sigma(\theta, R)} \|\lambda R(\lambda, A)\| := M < \infty,$ (9)

(ii) A and B are said to satisfy condition P if there are positive constants θ , θ' , R such that $\theta' < \theta < \pi/2$, A and B are of class $\Sigma(\theta + \pi/2, R)$, $\Sigma(\pi/2 - \theta', R)$, respectively.

If A and B are commuting operators, A + B is defined by (A + B)x = Ax + Bx with domain $D(A + B) = D(A) \cap D(B)$.

In this article we will use the following norm, defined by A on the space \mathbf{X} , $||x||_{\mathcal{T}_A} := ||R(\lambda, A)x||$, where $\lambda \in \rho(A)$. It is seen that different $\lambda \in \rho(A)$ yields equivalent norms. We say that an operator C on \mathbb{X} is A-closed if its graph is closed with respect to the topology induced by \mathcal{T}_A on the product $\mathbb{X} \times \mathbb{X}$. It is easily seen that C is A-closable if $x_n \to 0, x_n \in D(C), Cx_n \to y$ with respect to \mathcal{T}_A in \mathbb{X} implies y = 0. In this case, A-closure of C is denoted by $\overline{\mathcal{C}}^A$.

THEOREM 2 Assume that A and B commute. Then the following assertions hold:

(i) If one of the operators is bounded, then

$$\sigma(A+B) \subset \sigma(A) + \sigma(B). \tag{10}$$

(ii) If A and B satisfy condition P, then A + B is A-closable, and

$$\sigma\left(\overline{(A+B)}^{A}\right) \subset \sigma(A) + \sigma(B).$$
(11)

In particular, if D(A) is dense in \mathbb{X} , then $\overline{(A+B)}^A = \overline{A+B}$, where $\overline{A+B}$ denotes the usual closure of A+B.

Proof For the proof we refer the reader to [3, Theorems 7.2, 7.3].

2.4. Unbounded Perturbation of Operators

We will consider the Volterra equation (1) as a perturbed equation of the evolution equation u' = Au. To this end, the following perturbation theorem is the key tool.

THEOREM 3 Let A be of class $\Sigma(\theta, R)$ and M be defined by (9) for some $\pi > \theta > \pi/2$. Suppose also that B is a linear operator with $D(B) \supset D(A)$ and there are nonnegative reals α, β such that

$$||Bx|| \le \alpha ||Ax|| + \beta ||x||, \quad \forall x \in D(A).$$

There exists a positive number R' such that if $0 \le \alpha \le \delta := (1/2)(1+M)^{-1}$, then A + B is of class $\Sigma(\theta, R')$.

Proof The proof can be taken from the one in [18, the proof of Theorem 2.2.1, p. 80].

Remark 1 In this article we will consider the operator A which is of class $\Sigma(\theta + \pi/2, R)$ with $\pi/2 > \theta > 0$. In particular, if A is the generator of a uniformly bounded analytic semigroup of bounded linear operators, then A is of the above class.

3. BOUNDED AND ALMOST PERIODIC SOLUTIONS

From the standing hypothesis on $B(\cdot)$, it follows that if $u \in D(\mathcal{A}_{BUC(\mathbb{R},\mathbb{X})})$, then $u \in D(\mathcal{B}_{BUC(\mathbb{R},\mathbb{X})})$, where $\mathcal{B}_{BUC(\mathbb{R},\mathbb{X})}u(t) := \int_0^\infty dB(\eta)u(t-\eta)$. Moreover,

$$\|\mathcal{B}_{\mathrm{BUC}(\mathbb{R},\mathbb{X})}u(\cdot)\| \leq \mathrm{Var}B|_{0}^{\infty}\|\mathcal{A}_{\mathrm{BUC}(\mathbb{R},\mathbb{X})}u(\cdot)\| + \mathrm{Var}B|_{0}^{\infty}\|u(\cdot)\|, \quad \forall u \in D(\mathcal{B}_{\mathrm{BUC}(\mathbb{R},\mathbb{X})}).$$

We will fix a pair of nonnegative constants a, b (which may be different from the pair $(\operatorname{Var} B|_0^{\infty}, \operatorname{Var} B|_0^{\infty})$) such that

$$\|\mathcal{B}_{\text{BUC}(\mathbb{R},\mathbb{X})}u(\cdot)\| \le a\|\mathcal{A}_{\text{BUC}(\mathbb{R},\mathbb{X})}u(\cdot)\| + b\|u(\cdot)\|, \quad \forall u \in D(\mathcal{B}_{\text{BUC}(\mathbb{R},\mathbb{X})}).$$
(12)

Let $\Lambda \subset \mathbb{R}$ be a closed subset. We will consider the following abstract operator equation

$$\mathcal{D}u = \mathcal{A}u + \mathcal{B}u + f \tag{13}$$

where $\mathcal{D} := d/dt$, $\mathcal{A}u(t) = Au(t)$, $\mathcal{B}u(t) = \int_0^\infty dB(\tau)u(\tau - t)$ in the function space $\Lambda(\mathbb{X})$. Obviously, \mathcal{A} consists of all functions $w \in \Lambda(\mathbb{X})$ such that $w(t) \in D(\mathcal{A})$, $\forall t \in \mathbb{R}$ and $Aw(\cdot) \in BUC(\mathbb{R}, \mathbb{X})$, i.e., $\sup_{t \in \mathbb{R}} ||w(t)|| < \infty$ and $\sup_{t \in \mathbb{R}} ||Aw(t)|| < \infty$. $D(\mathcal{B})$ consists of all functions $w \in \Lambda(\mathbb{X})$ such that $w(t) \in D(\mathcal{A})$, $\forall t \in \mathbb{R}$ and if $v(t) := \int_0^{+\infty} dB(\eta)w(t - \eta)$, then $v \in BUC(\mathbb{R}, \mathbb{X})$. Hence, in $\Lambda(\mathbb{X})$, \mathcal{B} is \mathcal{A} -bounded, so it may be regarded as a unbounded perturbation of \mathcal{A} . Below we will consider the sum of the operators $-\mathcal{D}$ and $\mathcal{A} + \mathcal{B}$. These are commuting operators as shown in the following lemma.

LEMMA 2 Under the above notation, \mathcal{D} and $\mathcal{A} + \mathcal{B}$ are commuting operators.

Proof Obviously, $D(\mathcal{A} + \mathcal{B}) = D(\mathcal{A})$. To prove the lemma it suffices to show that for fixed λ_0, η_0 such that $\Re \lambda_0 > 0$ (hence $\lambda_0 \in \rho(\mathcal{D})$) and $\eta_0 \in \rho(\mathcal{A} + \mathcal{B})$ we have

$$R(\lambda_0, \mathcal{D})R(\eta_0, (\mathcal{A} + \mathcal{B})) = R(\eta_0, (\mathcal{A} + \mathcal{B}))R(\lambda_0, \mathcal{D}).$$
(14)

Recall that \mathcal{D} generates the translation group $(S(t))_{t \in \mathbb{R}}$. Hence,

$$R(\lambda_0, \mathcal{D})v = \int_0^\infty e^{-i\lambda_0 t} S(t)v \, dt, \quad \forall v \in \Lambda(\mathbb{X}).$$

Thus, it remains to show that the translation group $(S(t))_{t \in \mathbb{R}}$ commutes with $\mathcal{A} + \mathcal{B}$. In fact, let us denote $R(\eta_0, \mathcal{A} + \mathcal{B})f = w$. This means that w is the unique solution in $\Lambda(\mathbb{X})$ to the equation

$$\eta_0 w(t) - Aw(t) - \int_0^\infty dB(\mu) w(t-\mu) = f(t), \quad \forall t \in \mathbb{R}$$

for any fixed $f \in \Lambda(\mathbb{X})$. Of course, for every $\tau \in \mathbb{R}$ the function $S(\tau)w$ is a solution to the above equation with the right hand side $S(\tau)f \in \Lambda(\mathbb{X})$. Thus, $S(\tau)w = R(\eta_0, \mathcal{A} + \mathcal{B}) S(\tau)f$. From the arbitrary nature of $f \in \Lambda(\mathbb{X})$, this yields

$$S(\tau)R(\eta_0, \mathcal{A} + \mathcal{B}) = R(\eta_0, \mathcal{A} + \mathcal{B})S(\tau), \quad \forall \tau \in \mathbb{R},$$

i.e., the commutativeness of the translation group and the operator $R(\eta_0, A + B)$. This completes the proof of the lemma.

On the other hand, we have the following lemma.

LEMMA 3 Let A be of class $\Sigma(\theta + \pi/2, R)$ for $\pi/2 > \theta > 0$ with constant M and $a \le (1/2)(1 + M)^{-1}$, where a is a nonnegative real chosen in (12). Then on $\Lambda(\mathbb{X})$, $\mathcal{A} + \mathcal{B}$ is of class $\Sigma(\theta + \pi/2, R')$ with R' independent of Λ .

Proof The lemma is an immediate consequence of Theorem 3 and (12). In fact, by Lemma 1 and (v) of Theorem 1, \mathcal{A} is of the same class as \mathcal{A} . Again, by (viii) of Theorem 1 and Theorem 3 the operator $\mathcal{A} + \mathcal{B}$ is of class $\Sigma(\theta + \pi/2, R')$ for some positive R' independent of Λ (which depends only on the constants a, b, M, determined by $\mathcal{A}, \mathcal{B}(\cdot)$).

3.1. Definition of Admissibility

Definition 4

- (i) A function $u \in BUC(\mathbb{R}, \mathbb{X})$ is said to be a (*classical*) solution of Eq. (1) if $u'(\cdot) \in BUC(\mathbb{R}, \mathbb{X}), u(t) \in D(A), \forall t \in \mathbb{R}, Au(\cdot) \in BUC(\mathbb{R}, \mathbb{X})$ and Eq. (1) holds.
- (ii) (cf. [5, Def. 2.1, p. 41]) A function u ∈ BUC(ℝ, X) is said to be a mild solution of Eq. (1) if there are a sequences of classical solutions u_n(·) ∈ BUC(ℝ, X) of Eq. (1) with the right hand side f_n ∈ BUC(ℝ, X) such that u_n → u and f_n → f as n → ∞.

We adopt the definition of admissibility (see also [20]) as follows:

Definition 5 The function f in Eq. (1) is said to be *regular* if Eq. (1) has at least one classical solution u. A translation invariant closed subspace $\mathcal{M} \subset \text{BUC}(\mathbb{R}, \mathbb{X})$ is said to be *almost admissible* for Eq. (1) if for every $f \in \mathcal{M}$ Eq. (1) has a unique mild solution $u_f \in \mathcal{M}$ such that the operator $\mathcal{M} \ni f \mapsto u_f \in \mathcal{M}$ is continuous. An almost admissible function space \mathcal{M} is said to be *admissible* if every trigonometric monomial $xe^{i\lambda t}$, as a function of t, is regular if it belongs to \mathcal{M} .

Remark 2

(i) In the case where A is the generator of a strongly continuous semigroup and B(t) is extendable to a bounded linear operator on the whole space X such that $B(\cdot)$ is a function of bounded variation from $\mathbb{R}_+ \to L(\mathbb{X})$, the notion of admissibility means that for every $f \in \mathcal{M}$ there is a unique mild solution $x_f \in \mathcal{M}$ in the usual

sense to Eq. (1) (see [16, $\S3.4$]). In this case we will say that *B* is of *the bounded case*.

(ii) The notion of admissibility in [19] is a little stronger than the one in the above definition. In fact, in [20] it is required that all elements of M ∩ BUC¹(ℝ, X) be regular. This requirement yields that every trigonometric monomial xe^{λt} is regular if it is in M. However, if B is of the bounded case, all these notions are the same for the function space Λ(X) (see [16, Corollary 3.1]).

3.2. Necessary and Sufficient Conditions for Admissibility

Below we will denote

$$\Lambda_0 := \left\{ \lambda \in \mathbb{R} : \not \exists \left(i\lambda - A - \int_0^\infty dB(\eta) e^{-i\lambda\eta} \right)^{-1} \in L(\mathbb{X}, \mathbb{Y}) \right\}.$$

PROPOSITION 1 Let $\Lambda \in \mathbb{R}$ be a closed subset and $\Lambda(\mathbb{X})$ be admissible for Eq. (1). Then the following assertions hold:

- (i) $\Lambda \cap \Lambda_0 = \emptyset$,
- (ii) $\operatorname{sp}(u) \subset \Lambda_0$ for every (classical) solution u of the homogeneous equation (i.e. Eq. (1) with f = 0).

Proof The second assertion has been proved in [19, Proposition 1]. For the first assertion we note that translations commute with the Green operator G which takes every $f \in \Lambda(\mathbb{X})$ into the unique mild solution $u_f \in \Lambda(\mathbb{X})$. In fact, for classical solutions, this claim is obvious because if u_f is a classical solution of Eq. (1) with f, then $S(h)u_f$ is the unique classical solution of Eq. (1) with S(h)f for any $h \in \mathbb{R}$. Hence, S(h)Gf = GS(h)f. Recall that there is a dense subset of $\Lambda(\mathbb{X})$ of such regular functions f. Since the Green operator, by definition, is a (unique) extension of the operator $f \mapsto u_f$ defined on the set of regular f of $\Lambda(\mathbb{X})$, this operator commutes with translations as well. Hence, for the function $f(t) := xe^{i\lambda t}$, $x \in \mathbb{X}$, $\lambda \in \Lambda$ we have S(h)Gf = GS(h)f, $\forall h \in \mathbb{R}$. This yields that

$$\frac{dGf}{dt} = \lim_{h \to 0} \frac{S(h)Gf - Gf}{h} = \lim_{h \to 0} G \frac{S(h)f - f}{h} = i\lambda Gf.$$

This shows that $Gf(t) = ye^{i\lambda t}$ for some $y \in X$. On the other hand, by definition of admissibility, this trigonometric monomial should be a classical solution of Eq. (1). This yields in particular that $y \in Y$. Hence, we have proved the existence of an operator $P : X \ni x \mapsto y \in Y$. Obviously, by definition of admissibility, this operator is continuous. Next, we see that *P* is the bounded inverse of the following operator

$$Q: \mathbb{Y} \ni y \mapsto \left(i\lambda - A - \int_0^\infty dB(\xi)e^{-i\xi}\right) y \in \mathbb{X}.$$

In fact, both P, Q are continuous linear operators. Moreover, Q is injective, and obviously, $PQ = I_{\mathbb{Y}}, QP = I_{\mathbb{X}}$. Thus, $\lambda \notin \Lambda_0$.

Remark 3 Similar results were proved in [19, Proposition 1] for a little different definition of admissibility. Without additional conditions we cannot say anything about the relation between sp(u), sp(f), and Λ_0 .

PROPOSITION 2 Let A be of class $\Sigma(\theta + \pi/2, R)$ with $\pi/2 > \theta > 0$, B be of bounded variation with $a < (1/2)(1 + M)^{-1}$, where the constant a is chosen to satisfy (12) and M is determined by A. Then $\Lambda \cap \sigma_i(A + B) = \emptyset$ is a sufficient condition for the almost admissibility of the function space $\Lambda(\mathbb{X})$ for Eq. (1).

Proof The theorem is an immediate consequence of Theorem 2 and Lemma 3.

COROLLARY 1 Given A, B as in Proposition 2. Then if for sufficiently large N such that $\Lambda \cap [-N, N] = \emptyset$, $\Lambda(\mathbb{X})$ is almost admissible for Eq. (1).

Proof Under the corollary's assumptions, on the space $\Lambda'(\mathbb{X})$ the operator $\mathcal{A} + \mathcal{B}$ is of class $\Sigma(\theta + \pi/2, R')$ with $\pi/2 > \theta > 0$ with some R' independent of Λ' . This yields in particular that the set $\sigma_i(\mathcal{A} + \mathcal{B}) \subset [-R', R']$ for any closed subset $\Lambda' \subset \mathbb{R}$. Consequently, if $\Lambda \cap [-R', R] = \emptyset$, then in $\Lambda(\mathbb{X})$ all conditions of Proposition 2 are satisfied.

In this subsection we will prove the following theorem which is the main result of the article.

THEOREM 4 Let A be the generator of an analytic semigroup, and B be of bounded variation with $a < (1/2)(1 + M)^{-1}$, where the constant a is chosen to satisfy (12) and M is determined by A. Then, the function space $\Lambda(X)$ is admissible for Eq. (1) if and only if $\Lambda \cap \Lambda_0 = \emptyset$.

Proof It may be seen that it suffices to prove the theorem with the assumption that A is of class $\Sigma(\theta + \pi/2, R)$ for some $0 < \theta < \pi/2$ and R > 0. We will use the principle of superposition, i.e., if the equations $\dot{x} = Ax(t) + Bx(t) + f_1(t)$ and $\dot{y} = Ay(t) + By(t) + f_2(t)$ have two solutions $x(\cdot), y(\cdot)$. Then, the equation $\dot{x} = Ax(t) + Bx(t) + (f_1(t) + f_2(t))$ have a solution $x(\cdot) + y(\cdot)$. Now using the sums of the commuting operators method we can show that on $\Lambda(X)$ the operator $\mathcal{A} + \mathcal{B}$ is of class $\Sigma(\theta, R)$ for some positive R, $\pi > \theta > \pi/2$ (these constants are determined by the operator A, independent of Λ), so $\sigma_i(\mathcal{A} + \mathcal{B}) \subset [-R, R]$ (recall that $\sigma_i(\mathcal{A} + \mathcal{B}) := \{\xi \in \mathbb{R}: i\xi \in \sigma(\mathcal{A} + \mathcal{B})\}$. On the other hand, for every $f \in \Lambda(X)$ we can decompose it into two components one of which has spectrum outside the area of $\sigma_i(\mathcal{A} + \mathcal{B}) \subset [-R, R]$, say, f_2 . The other component f_1 has compact spectrum in [-R - 1, R + 1]. In fact, we can choose a continuous function φ on \mathbb{R} such that $\mathcal{F}\varphi(t) = 1$, $\forall t \in [-R, R]$, $\mathcal{F}\varphi(t) = 0$, $\forall |t| \ge R + 1$ and $0 \le \mathcal{F}\varphi(t)$, $\forall t \in \mathbb{R}$ (recall that $\mathcal{F}\varphi$ denotes the Fourier transform of φ). In this way, we can define $f_1 := \varphi * f$, $f_2 := f - \varphi * f$. By (vi) and (vii) of Theorem 1

$$\operatorname{sp}(f_1) \subset \operatorname{sp}(f) \cap \operatorname{supp}(\mathcal{F}\varphi) \subset \operatorname{sp}(f) \cap [-R-1, R+1]$$

$$\operatorname{sp}(f_2) \subset \operatorname{sp}(f) \cap \operatorname{supp}(1 - \mathcal{F}\varphi) \subset \operatorname{sp}(f) \cap \mathbb{R} \setminus [-R, R].$$

Thus, we can apply Prüss' result [19, Theorem 1] for the equation (1) with f_1 . The equation with f_2 can be solved by using Corollary 1. Now we show that $\Lambda(\mathbb{X})$ is admissible. To this end, we will establish the Green operator on $\Lambda(\mathbb{X})$ for Eq. (1) which will be defined on a dense subset \mathcal{F} of regular functions. We will denote

$$\Lambda_1 := \Lambda \cap [-R - 1, R + 1]$$
$$\Lambda_2 := \Lambda \cap (\mathbb{R} \setminus [-R, R]).$$

Let us denote by \mathcal{F}_2 the subset of $\Lambda_2(\mathbb{X})$ such that for every $f_2 \in \mathcal{F}_2$ Eq. (1) has a unique classical solution in $\Lambda_2(\mathbb{X})$. In $\Lambda_2(\mathbb{X})$, the operator $\overline{(\mathcal{D} - \mathcal{A} - \mathcal{B})_{\Lambda_2(\mathbb{X})}}$ is invertible. In fact, $\sigma(-\mathcal{D}_{\Lambda_2(\mathbb{X})}) = i\Lambda_2$, meanwhile, $i\sigma_i((\mathcal{A} + \mathcal{B})_{\Lambda_2(\mathbb{X})}) = \sigma((\mathcal{A} + \mathcal{B})_{\Lambda_2(\mathbb{X})}) \cap i\mathbb{R} \subset i[-R, R]$. Hence, Theorem 3 applies. Thus, \mathcal{F}_2 is dense in $\Lambda_2(\mathbb{X})$. We define $\mathcal{F} := \Lambda_1(\mathbb{X}) + \mathcal{F}_2$. Obviously, by the above argument, for every $f \in \mathcal{F}$ there is at least a classical solution $u_f \in \Lambda(\mathbb{X})$ to Eq. (1). We now show that such a solution u_f is unique. In fact, this is an immediate consequence of the assertion (ii) of Proposition 1. For every function $g \in \Lambda(X)$ we denote by $g_1 := \varphi * g, g_2 := g - \varphi * g$. Note that since $\mathcal{F}\varphi$ has compact support, this decomposition is continuous with respect to $g \in \Lambda(\mathbb{X})$. Solving Eq. (1) separately with respect to f_1 , f_2 we have solutions $G_1 f_1 := u_{f_1}, G_2 f_2 := u_{f_2}$, where G_1, G_2 denote the Green operator of Eq. (1) on $\Lambda_1(X)$ and the operator $((\overline{\mathcal{D}-\mathcal{A}-\mathcal{B}})_{\Lambda_2(X)})^{-1}$ defined on $\Lambda_2(X)$. The above argument shows that the mapping $G: \mathcal{F} \ni f \mapsto u_f := u_{f_1} + u_{f_2} \in \Lambda(\mathbb{X})$ can be extended uniquely to a continuous mapping on the whole space $\Lambda(\mathbb{X})$ by the formula $Gf := G_1 f_1 + G_2 f_2$. It remains to show that every trigonometric monomial $xe^{i\lambda t}$ is a regular function. In fact, since $\lambda \notin \Lambda_0$, $(i\lambda - A - \widehat{dB}(\lambda))^{-1} \in L(\mathbb{X}, \mathbb{Y})$. Hence, $ye^{i\lambda t}$ is a classical solution of Eq. (1) with $f(t) := xe^{i\lambda t}$, where $y := (i\lambda - A - \widehat{dB}(\lambda))^{-1}x$.

In the same way as above we consider the following equation

$$\dot{u}(t) = Au(t) + \int_{-\infty}^{+\infty} dB(\eta)u(t-\eta) + f(t),$$
(15)

where A is the generator of an analytic semigroup, $B(t) \in L(X)$, $\forall t \in \mathbb{R}$, and $B(\cdot)$ is of bounded variation, and f is a bounded continuous function. A minor modification of the proof of [19, Theorem 1] shows that its statement remains true for equations of the form (15). Applying exactly the method of proving Theorem 4 we arrive at the following result which resolves completely the bounded solution problem for the bounded case of B.

THEOREM 5 Let the above conditions be fulfilled for Eq. (15) and Λ be a closed subset of \mathbb{R} . Then the function space $\Lambda(\mathbb{X})$ is admissible for Eq. (15) if and only if $\Lambda \cap \Lambda_0 = \emptyset$.

Remark 4 Several particular cases of Theorem 5 has been proved in [11,12,15,16].

For a closed subset $\Lambda \subset \mathbb{R}$ let us denote by $AP_{\Lambda}(\mathbb{X})$ the subspace of $AP(\mathbb{X})$ consisting of all almost periodic functions f such that $sp(f) \subset \Lambda$. Obviously, $AP_{\Lambda}(\mathbb{X})$ is a closed subspace of $\Lambda(\mathbb{X})$ which is invariant under translations.

THEOREM 5 Let the conditions of Theorem 5 (Theorem 5, respectively) be satisfied. Then $AP_{\Lambda}(X)$ is admissible for Eq. (1) (Eq. (15), respectively).

Proof Note that the proof of [19, Theorem 1] is also applicable to the case of $AP_{\Lambda}(X)$. On the other hand, the statement of the results in Corollary 1 is also true for the case of $AP_{\Lambda}(X)$. Combining these facts and following the proof of Theorem 5 we get the theorem.

4. PERIODIC AND QUASI PERIODIC SOLUTIONS

To illustrate applications of the main results obtained in the previous section we consider below the admissibility of the function spaces of periodic and, more generally, quasi-periodic functions for Eq. (1).

4.1. Periodic Solutions

We consider the periodic solutions of Volterra equations of the form (1). As we can characterize the τ -periodicity of a bounded continuous function f by $\operatorname{sp}(f) \subset 2\pi \mathbb{Z}/\tau$, we have the following

COROLLARY 2 Let A be of class $\Sigma(\theta, R)$ for $\pi > \theta > \pi/2$ with constant M defined by (9), and B satisfy the standing hypothesis. Moreover, let $a < \delta := (1/2)(1 + M)^{-1}$. Then a necessary and sufficient condition for the space of τ -periodic continuous functions to be admissible for Eq. (1) is $\Lambda_0 \cap 2\pi \mathbb{Z}/\tau = \emptyset$.

4.2. Quasi Periodic Solutions

A set of reals S is said to have an *integer and finite basis* if there is a finite subset $T \subset S$ such that any element $s \in S$ can be represented in the form $s = n_1b_1 + \cdots + n_mb_m$, where $n_j \in \mathbb{Z}$, $j = 1, \ldots, m$, $b_j \in T$, $j = 1, \ldots, m$. If f is quasi-periodic, i.e., it is of the form $f(t) = F(t, t, \ldots, t)$, $t \in \mathbb{R}$, where $F(t_1, t_2, \ldots, t_n)$ is a X-valued continuous function of n variables which is periodic in each variable, the set of its Fourier–Bohr exponents is discrete (which coincides with sp(f) in this case), then the spectrum sp(f) has an integer and finite basis (see [13, p. 48]). Conversely, if f is almost periodic and sp(f) has an integer and finite basis, then f is quasi-periodic. We refer the reader to [13, pp. 42–48] for more information on the relation between quasi-periodicity and spectrum, Fourier–Bohr exponents of almost periodic functions.

COROLLARY 3 Let A be of class $\Sigma(\theta, R)$ for $\pi > \theta > \pi/2$ with constant M defined by (9), B satisfy the standing hypothesis, and $a < \delta := (1/2)(1 + M)^{-1}$. Moreover, let f be quasiperiodic with sp(f) having an integer and finite basis such that $sp(f) \cap \Lambda_0 = \emptyset$. Then Eq. (1) has a unique quasi-periodic mild solution x_f such that $sp(x_f) \subset sp(f)$.

Proof Under the assumptions, Theorem 5 can be applied to the function space $AP_{\Lambda}(X)$, where $\Lambda := sp(f)$. This shows that Eq. (1) has a unique mild solution x_f which is almost periodic and has the spectrum $sp(x_f) \subset sp(f)$. Since sp(f) has an integer and finite basis, so does $sp(x_f)$. But this fact yields that the almost periodic function x_f is quasi-periodic.

5. APPLICATIONS

Example 1 Consider the equation

$$x'(t) = Ax(t) + \sum_{k=1}^{n} A_k x(t - \tau_k) + f(t),$$

where $x(t) \in \mathbb{X}$, A is the generator of an analytic semigroup of bounded linear operators on a given Banach space \mathbb{X} , $A_k \in L(\mathbb{X})$, k = 1, 2, ..., n, $\tau_k \in \mathbb{R}$, 0, k = 1, 2, ..., n. In [11,12] a necessary and sufficient condition for the existence of ω -periodic solution has been given in the finite dimensional case. Here we consider the case of quasiperiodic solutions. For instance, $f(t) := g(t) + g(\sqrt{2}t)$, where g is a ω -periodic function. A necessary and sufficient condition for the equation to have a unique quasi-periodic solution x_f such that $sp(x_f) \subset ((2\pi \mathbb{Z}/\omega) + (2\sqrt{2\pi}\mathbb{Z}/\omega))$ for every f of the above form is

$$\left(\frac{2\pi\mathbb{Z}}{\omega} + \frac{2\sqrt{2}\pi\mathbb{Z}}{\omega}\right) \cap \Lambda_0 = \emptyset,$$

where $\Lambda_0 := \{\eta \in \mathbb{R} : \exists (i\eta I - \sum_{k=1}^n A_k e^{-i\tau_k \eta})^{-1} \in L(\mathbb{X})\}$. In particular if $\Lambda_0 = \emptyset$ we get the conclusion in [15, Section 4] for the existence and uniqueness of bounded solutions in the finite dimensional case.

Example 2 Consider the heat equation in materials with memory

$$u_t(t,x) = \Delta u(t,x) + \int_0^\infty \Delta dB(\tau)u(t-\tau) + \sum_{k=1}^n a_k u(t-\tau_k, x) + f(t,x), \quad t \in \mathbb{R}, \ x \in \Omega,$$
(16)

$$u(t, x) = 0, \quad t \in \mathbb{R}, \ x \in \partial\Omega, \tag{17}$$

where $\Omega \subset \mathbb{R}^N$ denotes a bounded domain with smooth boundary $\partial\Omega$ and $B \in BV(\mathbb{R}_+)$ and is left continuous, B(0+) = 0 and a_k , k = 1, 2, ... are reals. The case where $dB(\tau) = b(\tau)d\tau$ for a function $b(\cdot) \in L^1(\mathbb{R}_+)$ and $a_k = 0, k = 1, 2..., n$ and the case where both

$$a := \operatorname{Var} B|_0^\infty$$
 and $b := \sum_{k=1}^n ||a_k||$

are small enough have been treated in [20, Applications (a)]. We define $\mathbb{X} := L^2(\Omega)$, $A = \Delta$ with $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}$. Then, A is a self-adjoint negative definite hence analytic, and $\sigma(A) = {\{\mu_j\}}_1^\infty \subset (-\infty, 0)$. To apply Proposition 2, Theorems 4, 5 we assume that $\operatorname{Var} B|_0^\infty$ is sufficiently small. Meanwhile, the coefficients a_k can be arbitrarily large.

Example 3 Consider the following system of equations arising in population dynamics

$$u_{jt} = d_j \Delta u_j(t, x) + \sum_{k=1}^n \int_0^\infty db_{jk}(\tau) u_k(t - \tau, x), \quad t \in \mathbb{R}, \ x \in \Omega$$
$$d_j \frac{\partial u_j}{\partial \nu}(t, x) = 0, \quad x \in \partial\Omega, \ j = 1, \dots, n,$$

where $\Omega \subset \mathbb{R}^n$ denotes a bounded domain with smooth boundary and $\nu(x)$ the outer normal at $x \in \partial \Omega$. Let $\mathbb{X} := [L^2(\Omega)]^n$, $A := D \cdot \Delta = (\text{diag } d_j)\Delta$, $d_j \ge 0$ for all j with domain

$$D(A) := \{ u \in \mathbb{X} : d_i u_j \in W^{2,2}(\Omega), d_j \partial u_j / \partial \nu = 0 \text{ on } \partial \Omega \}$$

and let B(t) be defined as follows

$$(B(t)u)_j(x) = \sum_{k=1}^n b_{jk}(t)u_k(x), \quad u \in \mathbb{X},$$

where $b_{jk} \in BV(\mathbb{R}_+)$. Then *A* is the generator of an analytic semigroup and Theorem 5 is applicable without any assumption on the smallness of $VarB|_0^{\infty}$. As shown in [19, Applications], the spectrum of the equation can be computed as follows:

$$\Lambda_0 = \{ \rho \in \mathbb{R} : \kappa_m(i\rho) = 0 \text{ for some } m \in \mathbb{N}_0 \},\$$

where

$$\kappa_m = \det \Big(\lambda - \mu_m D - \widehat{dB}(\lambda) \Big), \quad m \in \mathbb{N}_0.$$

Here $\mu_0 := 0 > \mu_1 \ge \mu_2, \ldots$ denote the eigenvalues of the Laplacian with Neumann boundary condition.

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