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Periodic Solutions of Infinite Delay Evolution Equations

James H. Liu

Department of Mathematics, James Madison University, Harrisonburg, Virginia 22807
E-mail: Liujh@jmu.edu

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For $A(t)$ and $f(t, x, y)$ T-periodic in t, we consider the differential equation with infinite delay in a general Banach space X,

$$u'(t) + A(t)u(t) = f(t, u(t), u_t), \qquad t > 0, \qquad u(s) = \phi(s), \qquad s \le 0, \quad (0.1)$$

where the resolvent of the unbounded operator $A(t)$ is compact and f is continuous in its variables, and $u_t(s) = u(t + s)$, $s \le 0$. We first show that the Poincaré operator given by $P(\phi) = u_T(\phi)$ (i.e., T units along the unique solution $u(\phi)$ determined by the initial function ϕ) is a condensing operator with respect to Kuratowski's measure of non-compactness in a phase space C_g, and then derive periodic solutions from bounded solutions by using Sadovskii's fixed point theorem. This extends the study of deriving periodic solutions from bounded solutions to infinite delay differential equations in general Banach spaces. © 2000 Academic Press

Key Words: periodic solutions; infinite delay; Kuratowski's measure of noncompactness; Sadovskii's fixed point theorem.

1. INTRODUCTION

This paper is concerned with deriving periodic solutions from bounded solutions for the infinite delay differential equation

$$u'(t) + A(t)u(t) = f(t, u(t), u_t), \qquad t > 0, \qquad u(s) = \phi(s), \qquad s \le 0, \tag{1.1}$$

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in a general Banach space $(X, \|\cdot\|)$, with $A(t)$ a unbounded operator and f a continuous function in its variables, and $A(t)$ and $f(t, x, y)$ T-periodic in t. Here $u_t \in C((-\infty, 0], X)$ (space of continuous functions on $(-\infty, 0]$ with values in X) is defined by $u_t(s) = u(t+s)$, $s \leq 0$.

A standard approach in deriving T periodic solutions is to define the Poincaré operator [1] given by $P(\phi) = u_T(\phi)$, which maps an initial function (or value) T-units along the unique solution $u(\phi)$ determined by the initial function (or value) ϕ. Then conditions are given such that some fixed point theorem can be applied to get a fixed point for the Poincaré operator, which gives rise to a periodic solution.

Many fixed point theorems require that the operator maps among compact sets, or that the operator itself is compact; see, e.g., Browder's, Horn's, Schauder's and Schauder–Tychonov's fixed point theorems.

When X is a finite dimensional space, compact sets can be constructed using the uniformly bounded and equicontinuous functions. See, e.g., Burton [2] for such a construction in a phase space C_g, where the boundedness and ultimate boundedness are then used to ensure that Horn's fixed point theorem can be applied to obtain a fixed point for the Poincaré operator.

When X is a general (infinite dimensional) Banach space, then to verify the compactness of a set of functions with values in X, one needs to use the abstract version of the Ascoli theorem. Now, the additional requirement is to ensure that at any given point in the interval where the functions are defined, the set of all functions evaluated at that point is precompact in X. This requirement is very hard to fulfill, so it causes major difficulty for the study of periodic solutions in general Banach spaces.

For differential equations without delay or with finite delay in general Banach spaces, Amann [1], Hale [6], Xiang and Ahmed [17], Liu [10–12], etc., studied the existence of periodic solutions by requiring that the resolvent of $A(\cdot)$ be compact, so that the abstract version of the Ascoli theorem can be used to show that the Poincaré operator is compact. Hence the images of the Poincaré operator on bounded sets are precompact, which makes it possible to derive the periodic solutions. For example, for finite delay differential equations with initial functions on $[-r, 0]$, this is done in Liu [12] by observing that if the period $T > r$, then the image of an initial function under the Poincaré operator becomes a segment on $[T-r, T] \subset (0, \infty)$ of a smooth function defined on $(0, \infty)$. That means the (possibly) "bad" history of the initial function on $[-r, 0]$ has been cut, or smoothed out. Therefore, it is possible to show that the Poincaré operator is compact and hence to derive periodic solutions from bounded solutions.

However, this technique of showing the compactness of the Poincaré operator does not apply to differential equations with infinite delay in general Banach spaces. The reason is that now the (time) interval in the phase space is $(-\infty, 0]$, so that under the Poincaré operator, an initial function on $(-\infty, 0]$ becomes a segment on $(-\infty, 0]$ of a function defined on $(-\infty, T]$. That is, the history of the initial function on $(-\infty, 0]$ is still carried over under the Poincaré operator; thus it is possible that under the Poincaré operator, a bounded set gets mapped into a non-precompact set. Therefore the abstract version of the Ascoli theorem and hence all those Browder's, Horn's, Schauder's, and Schauder–Tychonov's fixed point theorems are not applicable to the Poincaré operator in the infinite delay cases in general Banach spaces.

This means that other methods are needed to study the periodic solutions for differential equations with infinite delay in general Banach spaces. When $A(t)$ is independent of t and when Eq. (1.1) is an integrodifferential equation, the periodic solutions are studied in Burton and Zhang [3] using Granas's degree theory and in Grimmer and Liu [5] using the limiting equation technique (when Eq. (1.1) is also linear).

Recently, Henriquez [7] studied the periodic solutions for Eq. (1.1) (when $A(t)$ is independent of t) in a seminormed abstract space with the axioms for abstract infinite delay differential equations introduced in Hale and Kato [8], and Kuratowski's measure of non-compactness is used to show that the Poincaré operator is condensing under some conditions, so that by Sadovskii's theorem [16], fixed points exist when a condensing operator maps a convex, closed, and bounded set into itself.

In this paper we will adopt the idea of [7] and study Eq. (1.1) in continuous functions space $C((-\infty, 0], X)$. We will choose a function g on $(-\infty, 0]$ in such a way that in the "weighted" (or "friendly" in some literature) phase space C_g, the Poincaré operator is shown to be condensing without those conditions and axioms imposed in [7].

Note that Sadovskii's fixed point theorem requires that the Poincaré operator maps a bounded set into itself. Thus some kind of boundedness of the solutions is required here. We will show that if solutions of Eq. (1.1) are bounded (even locally on $[0, T]$) in a strict sense, then Eq. (1.1) has T periodic solutions (see Definition 4.1 and Theorem 4.3). This way, the idea of deriving periodic solutions from bounded solutions can be extended to infinite delay differential equations in general Banach spaces.

We will study the solutions of Eq. (1.1) in Section 2, Kuratowski's measure of non-compactness in Section 3, and the periodic solutions in Section 4.

2. THE SOLUTIONS

In this section we study the existence and uniqueness of solutions for Eq. (1.1). We make the following assumptions.

ASSUMPTION 2.1. For a constant $T > 0$, $f(t+T, x, y) = f(t, x, y)$, $A(t+T) = A(t)$, $t \geq 0$. f is continuous in its variables and is locally Lipschitzian in the second and the third variables, and f maps a bounded set into a bounded set.

ASSUMPTION 2.2 [14, p. 150]. For $t \in [0, T]$ one has

(H1) The domain $D(A(t)) = D$ is independent of t and is dense in X.

(H2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $\operatorname{Re} \lambda \leq 0$ and is compact, and there is a constant M independent of λ and t such that

$$\|R(\lambda, A(t))\| \leq M(|\lambda| + 1)^{-1}, \qquad \operatorname{Re} \lambda \leq 0.$$

(H3) There exist constants L and $0 < a \leq 1$ such that

$$\|(A(t) - A(s))A(r)^{-1}\| \leq L|t - s|^{a}, \qquad s, t, r \in [0, T].$$

Under these assumptions, the results in, e.g., Amann [1], Friedman [4], Kielhöfer [9], and Pazy [14] imply the existence of a unique evolution system $U(t, s)$, $0 \leq s \leq t \leq T$ for Eq. (1.1). See [1, 4, 9, 14] for details.

Now, we define the phase space C_g for Eq. (1.1). First we have

LEMMA 2.1. *There exists an integer $K_0 > 1$ such that*

$$\left(\frac{1}{2}\right)^{K_0 - 1} M_0 < 1, \tag{2.1}$$

where $M_0 = \sup_{t \in [0, T]} \|U(t, 0)\|$ is finite. Next, let $w_0 = T/K_0$; then there exists a function g on $(-\infty, 0]$ such that $g(0) = 1$, $g(-\infty) = \infty$, g is decreasing on $(-\infty, 0]$, and for $d \geq w_0$ one has

$$\sup_{s \leq 0} \frac{g(s)}{g(s - d)} \leq \frac{1}{2}. \tag{2.2}$$

Proof. Such a function g exists; e.g., $g(s) = e^{-as}$ where $a > 0$ is such that $e^{aw_0} \geq 2$. ∎

For the function g given in Lemma 2.1, define the continuous functions space

$$C_g = \left\{ \phi \colon \phi \in C((-\infty, 0], X) \text{ and } \sup_{s \leq 0} \frac{\|\phi(s)\|}{g(s)} < \infty \right\}. \tag{2.3}$$

Then C_g coupled with the norm

$$|\phi|_g = \sup_{s \leq 0} \frac{\|\phi(s)\|}{g(s)}, \qquad \phi \in C_g, \tag{2.4}$$

is a Banach space [2]. Now we consider the existence and the uniqueness of solutions of Eq. (1.1).

THEOREM 2.1. *Let the Assumptions* 2.1, 2.2 *be satisfied, and let* $\phi \in C_g$ *be fixed. Then there exists a constant* $\alpha > 0$ *and a unique continuous function* $u \colon (-\infty, \alpha] \to X$ *such that* $u_0 = \phi$ *(i.e.,* $u(s) = \phi(s)$, $s \leq 0$*), and*

$$u(t) = U(t, 0)\phi(0) + \int_0^t U(t, h) f(h, u(h), u_h)\, dh, \qquad t \in [0, \alpha]. \tag{2.5}$$

Proof. We will use the contraction mapping theorem. With $\phi \in C_g$ being fixed and with $\alpha > 0$ yet to be determined, we consider the functions $u \in C((-\infty, \alpha], X)$ with $u_0 = \phi$ and define a map Q such that $(Qu)(s) = \phi(s)$ for $s \leq 0$; and for $t \in [0, \alpha]$,

$$(Qu)(t) = U(t, 0)\phi(0) + \int_0^t U(t, h) f(h, u(h), u_h)\, dh. \tag{2.6}$$

Using the property of the evolution system U, we have $Q \colon \{u \in C((-\infty, \alpha], X) \colon u_0 = \phi\} \to \{u \in C((-\infty, \alpha], X) \colon u_0 = \phi\}$. Next, for $u, v \in C((-\infty, \alpha], X)$ with $u_0 = v_0 = \phi$, one has for $t \in [0, \alpha]$,

$$\begin{aligned} (Qu)(t) - (Qv)(t) &= \int_0^t U(t, h)[f(h, u(h), u_h) - f(h, v(h), v_h)]\, dh \\ &= \int_0^t U(t, h)[f(h, u(h), u_h) - f(h, u(h), v_h)]\, dh \\ &\quad + \int_0^t U(t, h)[f(h, u(h), v_h) - f(h, v(h), v_h)]\, dh. \end{aligned} \tag{2.7}$$

Now, $U(t, h)$ is a bounded operator and f is locally Lipschitzian in the second and the third variable; and for $h \in [0, \alpha]$,

$$
\begin{aligned}
|u_h - v_h|_g &= \sup_{s \le 0} \frac{\|u_h(s) - v_h(s)\|}{g(s)} \le \sup_{s \le 0} \|u_h(s) - v_h(s)\| \\
&= \sup_{s \le 0} \|u(h+s) - v(h+s)\| \\
&= \sup_{s \in [-h, 0]} \|u(h+s) - v(h+s)\| \qquad (u(r) - v(r) = 0, r \le 0) \\
&\le \sup_{l \in [0, \alpha]} \|u(l) - v(l)\|; \qquad (2.8)
\end{aligned}
$$

thus, it is clear that we can obtain the result by using the contraction mapping theorem in $C([0, \alpha], X)$. Details will be left here. ∎

Note that the function u determined by Theorem 2.1 is often called "the mild solution of Eq. (1.1)." In our study here, we would like the Poincaré operator to be defined on the whole space C_g (i.e., solutions of Eq. (1.1) exist for all initial functions in C_g), so that in this paper "a solution of Eq. (1.1)" means a function u determined by Theorem 2.1, i.e., a mild solution of Eq. (1.1). This is also the case in [7] and in many other related papers. Also note that we are concerned with the periodic solutions here, so we may assume that solutions exist on $[0, \infty)$. We will write $u = u(\cdot, \phi)$ to indicate that u is the unique solution with the initial function ϕ.

Remark 2.1. To guarantee the uniqueness of the solutions, we assumed the local Lipschitzan condition for the function f. If the Lipschitzan condition is not assumed, then the existence of the solutions can still be obtained by using Schauder's fixed point theorem with an argument similar to the one in [7].

For the Banach space C_g defined above, we have

LEMMA 2.2. *Let u be a continuous function on $(-\infty, T]$ such that $|u_t|_g$ is finite for every $t \in [0, T]$. Then for any $0 \le h < r \le T$ with $r - h \ge w_0$ (w_0 is from Lemma* 2.1), *one has*

$$
|u_r|_g \le \max\left\{ \sup_{s \in [h, r]} \|u(s)\|, \frac{1}{2}|u_h|_g \right\}. \qquad (2.9)
$$

Proof. By using Lemma 2.1, we have

$$
\begin{aligned}
|u_r|_g &= \sup_{s\le 0}\frac{\|u_r(s)\|}{g(s)} = \sup_{s\le 0}\frac{\|u(r+s)\|}{g(s)}\\
&= \sup_{l\le r}\frac{\|u(l)\|}{g(l-r)} \qquad (r+s=l)\\
&\le \max\left\{\sup_{l\in[h,r]}\|u(l)\|\frac{1}{g(l-r)},\ \sup_{l\le h}\frac{\|u(l)\|}{g(l-r)}\right\}\\
&= \max\left\{\sup_{l\in[h,r]}\|u(l)\|\frac{1}{g(l-r)},\ \sup_{l\le h}\frac{\|u(l)\|}{g(l-h)}\,\frac{g(l-h)}{g(l-r)}\right\}\\
&\le \max\left\{\sup_{l\in[h,r]}\|u(l)\|,\ \sup_{s\le 0}\frac{\|u(h+s)\|}{g(s)}\,\frac{g(s)}{g(s-(r-h))}\right\}\\
&\qquad\qquad (l-h=s)\\
&\le \max\left\{\sup_{s\in[h,r]}\|u(s)\|,\ \frac{1}{2}|u_h|_g\right\}. \qquad (2.10)
\end{aligned}
$$

∎

To estimate the solutions, we have

LEMMA 2.3. *Let the Assumptions* 2.1, 2.2 *be satisfied and let u and y be two solutions of Eq.* (1.1) (*with initial functions u_0 and y_0, respectively*) *on* $(-\infty, L]$, $L > 0$. *Then for $t \in [0, L]$,*

$$|u_t - y_t|_g \le (M_0 + 1)|u_0 - y_0|_g e^{M_1(k_0+k_1)t}, \tag{2.11}$$

where M_0, M_1, k_0, and k_1 are some constants.

Proof. Similar to the proof of Lemma 2.2, we have, for t $\in [0, L]$,

$$|u_t - y_t|_g \le \max\left\{\sup_{s\in[0,t]}\|u(s) - y(s)\|, |u_0 - y_0|_g\right\}. \tag{2.12}$$

Next, using the local Lipschitz conditions, we may assume that

$$\|f(h, u(h), y_h) - f(h, y(h), y_h)\| \le k_0\|u(h) - y(h)\|, \tag{2.13}$$

$$\|f(h, u(h), u_h) - f(h, u(h), y_h)\| \le k_1|u_h - y_h|_g, \tag{2.14}$$

for some constants k_0 and k_1. Let $M_0 = \sup_{t \in [0,T]} \|U(t,0)\|$, $M_1 = \sup_{0 \le h \le s \le T} \|U(s,h)\|$, which are finite. Then for $s \in [0,t]$,

$$\begin{aligned}
\|u(s) - y(s)\| &= \|U(s,0)(u(0) - y(0)) \\
&\quad + \int_0^s U(s,h)[f(h,u(h),u_h) - f(h,y(h),y_h)]\, dh\| \\
&= \|U(s,0)(u(0) - y(0)) \\
&\quad + \int_0^s U(s,h)[f(h,u(h),u_h) - f(h,u(h),y_h)]\, dh\| \\
&\quad + \int_0^s U(s,h)[f(h,u(h),y_h) - f(h,y(h),y_h)]\, dh\| \\
&\le M_0 \|u(0) - y(0)\| + \int_0^s M_1 k_0 \|u(h) - y(h)\|\, dh \\
&\quad + \int_0^s M_1 k_1 |u_h - y_h|_g\, dh \\
&\le M_0 |u_0 - y_0|_g + \int_0^t M_1 (k_0 + k_1) |u_h - y_h|_g\, dh. \qquad (2.15)
\end{aligned}$$

Thus we have, from (2.12),

$$|u_t - y_t|_g \le (M_0 + 1)|u_0 - y_0|_g + \int_0^t M_1(k_0 + k_1)|u_h - y_h|_g\, dh. \qquad (2.16)$$

Now, Gronwall's inequality implies (2.11). ∎

An immediate consequence of Lemma 2.3 is the following local boundedness property of the solutions.

THEOREM 2.2. *Let the Assumptions* 2.1, 2.2 *be satisfied and let* $D \subset C_g$ *be bounded. Then for any* $L > 0$, *solutions of Eq.* (1.1) *with initial functions in* D *are bounded on* $[0, L]$. *That is, there exists a constant* $E = E(D, L) > 0$ *such that if* $u(\cdot) = u(\cdot, \phi)$ *with* $\phi \in D$, *then* $\|u(t)\| \le E$ *for* $t \in [0, L]$.

Proof. Let $y = y(\phi_0)$ be a fixed solution with $\phi_0 \in D$. Then $|u_L|_g \le |u_L - y_L|_g$, and hence Lemma 2.3 implies that $\{|u_L(\phi)|_g : \phi \in D\}$ is bounded. Therefore the result is true by using the definition of the norm in C_g. ∎

3. THE MEASURE OF NON-COMPACTNESS

In this section, we examine Kuratowski's measure of non-compactness, which will be used in the next section to study the periodic solutions via the fixed points of a condensing operator. Kuratowski's measure of non-compactness (or the α measure) for a bounded set H of a Banach space Y with norm $|\cdot|_Y$ is defined as

$$\alpha(H) = \inf\{d > 0 : H \text{ has a finite cover of diameter } < d\}. \qquad (3.1)$$

We need to use the following basic properties of the α measure and Sadovskii's fixed point theorem here; see [13, 16].

LEMMA 3.1 [13]. *Let A and B be bounded sets of a Banach space Y. Then*

1. $\alpha(A) \leq \operatorname{dia}(A)$. ($\operatorname{dia}(A) = \sup\{|x-y|_Y : x, y \in A\}$.)
2. $\alpha(A) = 0$ *if and only if A is precompact.*
3. $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \Re$. ($\lambda A = \{\lambda x : x \in A\}$.)
4. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.
5. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$. ($A + B = \{x + y : x \in A, y \in B\}$.)
6. $\alpha(A) \leq \alpha(B)$ *if $A \subseteq B$.*

LEMMA 3.2 (Sadovskii's fixed point theorem [16]). *Let P be a condensing operator on a Banach space Y, i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) < \alpha(B)$ for every bounded set B of Y with $\alpha(B) > 0$. If $P(H) \subseteq H$ for a convex, closed, and bounded set H of Y, then P has a fixed point in H.*

The following result is also needed here. However, we have not found a reference for the result yet. So we provide a proof here.

LEMMA 3.3. *Let A with norm $|\cdot|_A$ and C with norm $|\cdot|_C$ be bounded. If there is a surjective map Q: $C \to A$ such that for any $c, d \in C$ one has $|Q(c) - Q(d)|_A \leq |c - d|_C$, then $\alpha(A) \leq \alpha(C)$.*

Proof. For any $\varepsilon > 0$, there exist bounded sets $G^i \subseteq C$, $i = 1, \ldots, m$, such that

$$\operatorname{dia}(G^i) \leq \alpha(C) + \varepsilon, \qquad C = \bigcup_{i=1}^{m} G^i. \qquad (3.2)$$

Now, Q is surjective, so that $A = \bigcup_{i=1}^{m} Q(G^i)$. And for $a, b \in Q(G^i)$ we may assume that $a = Q(c)$, $b = Q(d)$ for some $c, d \in G^i$. Thus

$$\begin{aligned} |a - b|_A = |Q(c) - Q(d)|_A \le |c - d|_C \le \mathrm{dia}(G^i) \\ \le \alpha(C) + \varepsilon. \end{aligned} \tag{3.3}$$

This implies $\mathrm{dia}(Q(G^i)) \le \alpha(C) + \varepsilon$, and hence from Lemma 3.1(1), $\alpha(Q(G^i)) \le \mathrm{dia}(Q(G^i)) \le \alpha(C) + \varepsilon$. Therefore Lemma 3.1(4) implies that $\alpha(A) \le \alpha(C) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result is true. ∎

Next, for $D \subset C_g$ and $u(\phi)$, the unique solution with $u_0(\phi) = \phi$, we define $W_l(D) = \{u_l(\phi) : \phi \in D\}$ and $W_{[h,r]}(D) = \{u_{[h,r]}(\phi) : \phi \in D\}$, where $u_{[h,r]}$ means the restriction of u on $[h, r]$. The idea of the proof of the following result is similar to the one in [15].

LEMMA 3.4. *Let the Assumptions* 2.1, 2.2 *be satisfied. If* $D \subset C_g$ *is bounded, then* $W_{[0,T]}(D) \subset C([0,T], X)$ *is bounded and* $W_r(D) \subset C_g$ *is bounded for each* $r \in [0, T]$. *And for any* $0 \le h < r \le T$ *with* $r - h \ge w_0$ (w_0 *is from Lemma* 2.1), *one has*

$$\alpha\big(W_r(D)\big) \le \max\left\{\alpha\big(W_{[h,r]}(D)\big), \frac{1}{2}\alpha\big(W_h(D)\big)\right\}. \tag{3.4}$$

Proof. First, Theorem 2.2 implies that $W_{[0,T]}(D) \subset C([0,T], X)$ is bounded. This result and Lemma 2.2 (with $h = 0$) imply that for each $r \in [0, T]$, $W_r(D)$ is bounded in C_g. Now, for any $\varepsilon > 0$, there exist bounded sets $P^i \subseteq W_{[h,r]}(D)$, $i = 1, \ldots, m$, and bounded sets $Q^j \subseteq W_h(D)$, $j = 1, \ldots, n$, such that

$$\mathrm{dia}(P^i) \le \alpha\big(W_{[h,r]}(D)\big) + \varepsilon, \qquad W_{[h,r]}(D) = \bigcup_{i=1}^{m} P^i, \tag{3.5}$$

$$\mathrm{dia}(Q^j) \le \alpha\big(W_h(D)\big) + 2\varepsilon, \qquad W_h(D) = \bigcup_{j=1}^{n} Q^j. \tag{3.6}$$

Put

$$Y_r^{i,j} = \big\{u_r \in W_r(D) : u_{[h,r]} \in P^i, u_h \in Q^j\big\}. \tag{3.7}$$

Then we have

$$W_r(D) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} Y_r^{i,j}. \tag{3.8}$$

For each $Y_r^{i,j}$, if $u_r, w_r \in Y_r^{i,j}$, then from the proof of Lemma 2.2,

$$\begin{aligned}|u_r - w_r|_g &\le \max\left\{\sup_{s\in[h,r]} \|u(s) - w(s)\|, \frac{1}{2}|u_h - w_h|_g\right\} \\ &\le \max\left\{\operatorname{dia}(P^i), \frac{1}{2}\operatorname{dia}(Q^j)\right\} \\ &\le \max\left\{\alpha\big(W_{[h,r]}(D)\big) + \varepsilon, \frac{1}{2}\big(\alpha(W_h(D)) + 2\varepsilon\big)\right\} \\ &= \max\left\{\alpha\big(W_{[h,r]}(D)\big), \frac{1}{2}\alpha\big(W_h(D)\big)\right\} + \varepsilon. \end{aligned} \tag{3.9}$$

This implies, using Lemma 3.1(1), that

$$\alpha\big(Y_r^{i,j}\big) \le \operatorname{dia}\big(Y_r^{i,j}\big) \le \max\left\{\alpha\big(W_{[h,r]}(D)\big), \frac{1}{2}\alpha\big(W_h(D)\big)\right\} + \varepsilon. \tag{3.10}$$

Then Lemma 3.1(4) implies that

$$\alpha\big(W_r(D)\big) \le \max\left\{\alpha\big(W_{[h,r]}(D)\big), \frac{1}{2}\alpha\big(W_h(D)\big)\right\} + \varepsilon. \tag{3.11}$$

Since $\varepsilon > 0$ is arbitrary, the result is true. ∎

The following lemma from Amann [1] will be used here to show that P is a condensing operator. Recall that in the usual way (see, e.g., Amann [1], Friedman [4], Pazy [14]) we define fractional power operator A^α and Banach space X_α for $0 \le \alpha \le 1$, where $A = A(0)$ and $X_\alpha = (D(A^\alpha), \|\cdot\|_\alpha)$ with $\|x\|_\alpha \equiv \|A^\alpha x\|$. We also write the norm in $L(X_\alpha, X_\beta)$ (space of bounded linear operators from X_α to X_β) as $\|\cdot\|_{\alpha,\beta}$.

LEMMA 3.5 [1]. (i) *Suppose that* $0 \le \alpha \le \beta < 1$. *Then for* $\beta - \alpha < \gamma < 1$, *there is a constant* $C(\alpha, \beta, \gamma)$ *such that*

$$\|U(t,h)\|_{\alpha,\beta} \le C(\alpha,\beta,\gamma)(t-h)^{-\gamma}, \qquad 0 \le h < t \le T.$$

(ii) *For* $0 \le \gamma < 1$, *there is a constant* $C(\gamma)$, *such that for* $g \in C([0, L], X)$ ($L > 0$ *is a constant*), *one has for* $0 \le s, t \le L$,

$$\left\|\int_0^t U(t,h)g(h)\,dh - \int_0^s U(s,h)g(h)\,dh\right\| \le C(\gamma)|t-s|^\gamma \max_{0\le h\le L}\|g(h)\|.$$

(iii) *Let* $0 \le \alpha < \beta \le 1$. *Then*

$$K(x,g)(t) \equiv U(t,0)x + \int_0^t U(t,h)g(h)\,dh, \qquad 0 \le t \le T,$$

defines a continuous linear operator from $X_\beta \times C([0,T], X)$ *into* $C^\gamma([0,T], X_\alpha)$ *for every* $\gamma \in [0, \beta - \alpha)$.

We also state here the Ascoli theorem for a general Banach space X for convenient reference.

LEMMA 3.6 (Ascoli theorem). *Let* $E \subset C([a,b], X)$ *be bounded. Then* E *is precompact if and only if functions in* E *are equicontinuous and for each* $t \in [a,b]$, *the set* $\{f(t) : f \in E\}$ *is precompact in* X.

By using the Ascoli theorem, we have the following result. The idea of the proof is similar to the case for finite delay in [12].

LEMMA 3.7. *Let the Assumptions* 2.1, 2.2 *be satisfied and let* $D \subset C_g$ *be bounded. Then* $\alpha(W_{[l,r]}(D)) = 0$ *for any* $0 < l < r \le T$.

Proof. By Lemma 3.1(2), we need to prove that the Ascoli theorem can be applied to the bounded set $E = W_{[l,r]}(D) \subset C([l,r], X)$.

Note that a function in E can be expressed as, for $s \in [l,r]$,

$$u(s,\phi) = U(s,0)\phi(0) + \int_0^s U(s,h) f(h, u(h), u_h)\, dh, \qquad \phi \in D. \tag{3.12}$$

Since $l > 0$, there is $k > 0$ such that $s > k$ for $s \in [l,r]$. From [14, p. 164], one has, for $s \in [l,r]$,

$$U(s,0)\phi(0) = U(s,k)U(k,0)\phi(0), \qquad \phi \in D. \tag{3.13}$$

Fix $\eta \in (0,1)$. Then from Lemma 3.5(i), $U(k,0)$: $X \to X_\eta$ is bounded. Next the embedding $X_\eta \to X$ is compact (under Assumption 2.2(H2); see, e.g., [6]); thus $\{U(k,0)\phi(0) : \phi \in D\}$ is precompact in X since $\{\phi(0) : \phi \in D\}$ is bounded in X. Therefore, the closure of $\{U(k,0)\phi(0) : \phi \in D\}$ is compact in X. Now, by a standard argument, one can verify that as functions on $[l,r]$,

$$\{U(\cdot,0)\phi(0) : \phi \in D\} = \{U(\cdot,k)[U(k,0)\phi(0)] : \phi \in D\} \tag{3.14}$$

is equicontinuous. Next, from Lemma 3.5(ii), for $0 \le \gamma < 1$, there is a constant $C(\gamma)$, such that for $s_1, s_2 \in [l,r]$,

$$\left\| \int_0^{s_2} U(s_2,h) f(h,u(h),u_h)\, dh - \int_0^{s_1} U(s_1,h) f(h,u(h),u_h)\, dh \right\| \tag{3.15}$$

$$\le C(\gamma)|s_1 - s_2|^\gamma \max_{0 \le h \le T} \|f(h,u(h),u_h)\|. \tag{3.16}$$

By using Lemma 3.4, we see that the variables in f are bounded. Now f maps a bounded set into a bounded set; thus there exists $M_2 = M_2(D) > 0$ such that

$$\|f(t,u(t),u_t(\phi))\| \le M_2, \qquad t \in [0,T], \qquad \phi \in D.$$

Therefore, as functions on $[l, r]$,

$$\left\{\int_0^{\cdot} U(\cdot, h) f(h, u(h), u_h)\, dh : \phi \in D\right\} \tag{3.17}$$

is also equicontinuous. Therefore, functions in E are equicontinuous.

In the following, to check the precompactness of the functions at every point in $[l, r]$, we fix $s_0 \in [l, r]$. From the above arguments, we also know that

$$\{U(s_0, 0)\phi(0) : \phi \in D\} \tag{3.18}$$

is precompact in X. Next, for $\phi \in D$, we may let $g(t) = f(t, u(t), u_t(\phi))$. Then

$$\int_0^{s_0} U(s_0, h) f(h, u(h), u_h(\phi))\, dh \equiv K(0, g)(s_0) \in X_\eta$$

according to Lemma 3.5(iii). Also note that by Lemma 3.5(i), there are constants $\gamma \in (0, 1)$ and $M_3 > 0$ such that

$$\|U(s_0, h)\|_{0,\eta} \leq M_3(s_0 - h)^{-\gamma}, \qquad 0 \leq h < s_0.$$

Thus

$$\left\|\int_0^{s_0} U(s_0, h) f(h, u(h), u_h(\phi))\, dh\right\|_\eta \leq M_3 M_2 T^{1-\gamma}/(1-\gamma), \qquad \phi \in D.$$

Therefore

$$\left\{\int_0^{s_0} U(s_0, h) f(h, u(h), u_h(\phi))\, dh : \phi \in D\right\} \tag{3.19}$$

is bounded in X_η. Then, using the fact that the embedding $X_\eta \to X$ is compact again, we see that the set defined by (3.19) is precompact in X. Now the Ascoli theorem implies that E is precompact. Thus by Lemma 3.1(2), $\alpha(W_{[l,r]}(D)) = \alpha(E) = 0$. ∎

4. THE PERIODIC SOLUTIONS

In this section, we study the periodic solutions of the T periodic infinite delay differential equation (1.1). For this purpose, we define the Poincaré operator P along the solution such that for the unique solution $u(\cdot, \phi)$ of

Eq. (1.1) with the initial function ϕ,

$$P(\phi) = u_T(\cdot, \phi), \qquad \phi \in C_g \tag{4.1}$$
$$\left(\text{i.e., } (P\phi)(s) = u_T(s, \phi) = u(T + s, \phi), \qquad s \le 0\right),$$

and then examine when the map P has a fixed point. We note that a fixed point of P gives rise to a periodic solution, because if $P(\phi) = \phi$, then for the solution $u(\cdot) = u(\cdot, \phi)$ with $u_0(\cdot, \phi) = \phi$, we can define $y(t) = u(t + T)$. Now, for $t \ge 0$, we can use the known formulas [14] $U(t, s) = U(t, r)U(r, s)$, $0 \le s \le r \le t \le T$, and $U(t + T, s + T) = U(t, s)$ (since the operator $A(t)$ is T periodic in t) to obtain

$$\begin{aligned}
y(t) &= u(t + T) = U(t + T, 0)\phi(0) \\
&\quad + \int_0^{t+T} U(t + T, h) f(h, u(h), u_h)\, dh \\
&= U(t + T, T)U(T, 0)\phi(0) + \int_0^T U(t + T, h) f(h, u(h), u_h)\, dh \\
&\quad + \int_T^{t+T} U(t + T, h) f(h, u(h), u_h)\, dh \\
&= U(t, 0)U(T, 0)\phi(0) + \int_0^T U(t + T, T)U(T, h) f(h, u(h), u_h)\, dh \\
&\quad + \int_0^t U(t + T, T + s) f(T + s, u(T + s), u_{T+s})\, ds \\
&= U(t, 0)U(T, 0)\phi(0) + \int_0^T U(t, 0)U(T, h) f(h, u(h), u_h)\, dh \\
&\quad + \int_0^t U(t, s) f(s, y(s), y_s)\, ds \\
&= U(t, 0)\left[U(T, 0)\phi(0) + \int_0^T U(T, h) f(h, u(h), u_h)\, dh\right] \\
&\quad + \int_0^t U(t, s) f(s, y(s), y_s)\, ds \\
&= U(t, 0)u(T) + \int_0^t U(t, s) f(s, y(s), y_s)\, ds \\
&= U(t, 0)y(0) + \int_0^t U(t, s) f(s, y(s), y_s)\, ds.
\end{aligned} \tag{4.2}$$

Equation (4.2) implies that y is also a solution with $y_0 = u_T(\phi) = P(\phi) = \phi$; then the uniqueness implies that $y(t) = u(t)$, so that $u(t) = u(t+T)$ is a T periodic solution.

Next, we prove that the operator P is condensing.

THEOREM 4.1. *Let the Assumptions* 2.1, 2.2 *be satisfied. Then the operator P defined by* (4.1) *is condensing in C_g with g given in Lemma* 2.1.

Proof. From Lemma 2.3, we have

$$|P(\phi) - P(\varphi)|_g = |u_T(\phi) - u_T(\varphi)|_g \le (M_0 + 1)e^{M_1(k_0+k_1)T}|\phi - \varphi|_g, \tag{4.3}$$

so that P is continuous. It also implies that P takes bounded sets into bounded sets (see the proof of Theorem 2.2). Next, let $D \subset C_g$ be bounded with $\alpha(D) > 0$. By using Lemmas 3.4 and 3.7 repeatedly, we have (w_0 is from Lemma 2.1)

$$\begin{aligned}
\alpha(P(D)) &= \alpha(W_T(D)) \le \max\left\{\alpha\left(W_{[T-w_0,T]}(D)\right), \frac{1}{2}\alpha\left(W_{T-w_0}(D)\right)\right\} \\
&= \frac{1}{2}\alpha\left(W_{T-w_0}(D)\right) \\
&\le \frac{1}{2}\max\left\{\alpha\left(W_{[T-2w_0,T-w_0]}(D)\right), \frac{1}{2}\alpha\left(W_{T-2w_0}(D)\right)\right\} \\
&= \left(\frac{1}{2}\right)^2 \alpha\left(W_{T-2w_0}(D)\right) \\
&\le \left(\frac{1}{2}\right)^2 \max\left\{\alpha\left(W_{[T-3w_0,T-2w_0]}(D)\right), \frac{1}{2}\alpha\left(W_{T-3w_0}(D)\right)\right\} \\
&= \left(\frac{1}{2}\right)^3 \alpha\left(W_{T-3w_0}(D)\right) \\
&\cdots \\
&\le \left(\frac{1}{2}\right)^{K_0-1} \max\left\{\alpha\left(W_{[0,T-(K_0-1)w_0]}(D)\right), \frac{1}{2}\alpha(D)\right\}.
\end{aligned} \tag{4.4}$$

Next, for $\cdot \in [0, T-(K_0-1)w_0]$,

$$\begin{aligned}
W_{[0,T-(K_0-1)w_0]}(D) &\subseteq \{U(\cdot,0)\phi(0) : \phi \in D\} \\
&\quad + \left\{\int_0^{\cdot} U(\cdot,h) f(h,u(h),u_h(\phi))\,dh : \phi \in D\right\}.
\end{aligned} \tag{4.5}$$

And since for $t \in [0, T - (K_0 - 1)w_0]$,

$$\|U(t,0)\phi(0) - U(t,0)\varphi(0)\| = \|U(t,0)(\phi(0) - \varphi(0))\| \\ \leq M_0\|\phi(0) - \varphi(0)\| \leq M_0|\phi - \varphi|_g, \quad (4.6)$$

where $M_0 = \sup_{t\in[0,T]}\|U(t,0)\|$, then we have from Lemma 3.1(3) and Lemma 3.3 that (for $\cdot \in [0, T - (K_0 - 1)w_0]$)

$$\alpha\{U(\cdot,0)\phi(0) : \phi \in D\} \leq M_0\alpha(D). \quad (4.7)$$

Similar to the proof in Lemma 3.7 we see that for $\cdot \in [0, T - (K_0 - 1)w_0]$,

$$\alpha\left\{\int_0^{\cdot} U(\cdot,h)f(h,u(h),u_h(\phi))\,dh : \phi \in D\right\} = 0. \quad (4.8)$$

Therefore we have from Lemma 3.1(5,6) that

$$\alpha\left(W_{[0,T-(K-1)w_0]}(D)\right) \leq M_0\alpha(D). \quad (4.9)$$

Thus, from Lemma 2.1 and (4.4), (4.9) we have (note that $M_0 = \sup_{t\in[0,T]}\|U(t,0)\| \geq 1$)

$$\alpha(P(D)) \leq \left(\frac{1}{2}\right)^{K_0-1}\max\left\{M_0\alpha(D), \frac{1}{2}\alpha(D)\right\} \\ \leq \left(\frac{1}{2}\right)^{K_0-1}M_0\alpha(D) < \alpha(D). \quad (4.10)$$

∎

Next, we study the periodic solutions of Eq. (1.1). From Lemma 3.2 and Theorem 4.1, we have

THEOREM 4.2. *Let the Assumptions* 2.1, 2.2 *be satisfied and let the operator P be defined by* (4.1) *in C_g with g given in Lemma* 2.1. *If there exists a convex, closed, and bounded set $H \subset C_g$ such that $P(H) \subseteq H$, then P has a fixed point in H, and hence Eq.* (1.1) *has a T periodic solution.*

Note that, in general, $f(t,0,0) \neq 0$ in Eq. (1.1), so the periodic solutions (if they exist) are nontrivial. Also, note that Sadovskii's fixed point theorem requires that $P(H) \subseteq H$ for a bounded set H. Therefore some kind of boundedness of the solutions of Eq. (1.1) is required here. We now make the following definition for the T periodic infinite delay differential equation (1.1).

DEFINITION 4.1. Solutions of Eq. (1.1) are said to be locally strictly bounded if there exists a constant $B > 0$ such that $|\phi|_g \leq B$ implies that its solution satisfies $\|u(t, \phi)\| \leq B$ for $t \in [0, T]$.

We now study the relationship between the boundedness and the periodicity of the solutions of Eq. (1.1). We will see that if solutions of Eq. (1.1) are bounded (even locally on $[0, T]$) in a strict sense, then Eq. (1.1) has T periodic solutions.

THEOREM 4.3. *Let the Assumptions* 2.1, 2.2 *be satisfied. If the solutions of Eq.* (1.1) *are locally strictly bounded* (*or assume that solutions are non-increasing in norm* $\|\cdot\|$ *on* $[0, T]$), *then Eq.* (1.1) *has a* T *periodic solution.*

Proof. Let the operator P be defined by (4.1) in C_g with g given in Lemma 2.1, and let $H = \{\phi \in C_g : |\phi|_g \leq B\}$ with B from Definition 4.1. Then H is convex, closed, and bounded in C_g. Next, for $u(\cdot) = u(\cdot, \phi)$ with $\phi \in H$, the locally strict boundedness implies that $\|u(t)\| \leq B$ for $t \in [0, T]$. Then we obtain from Lemma 2.2 that

$$\begin{aligned} |P(\phi)|_g = |u_T(\phi)|_g &\leq \max\left\{ \sup_{s \in [0, T]} \|u(s)\|, \frac{1}{2}|\phi|_g \right\} \\ &\leq \max\left\{ B, \frac{1}{2}B \right\} = B. \end{aligned} \tag{4.11}$$

Thus the result is true by using Theorem 4.2. ∎

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