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Bounded and periodic solutions of finite delay evolution equations

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1. Introduction

Let us consider the following finite delay evolution equation

$$u'(t)+A(t)u(t)=f(t,u(t),u_t), \quad t>0, \quad u(s)=\phi(s), \ s\in[-r,0], \tag{1.1}$$

in a general Banach space $(X,\|\cdot\|)$, with $A(t)$ an unbounded operator and f a continuous function. Here $r>0$ is a constant. (When $r=0$, (1.1) is an equation without delay.) We denote $C([-r,0],X)$ the space of continuous functions from $[-r,0]$ to X with the sup-norm, $\|\phi\|_C=\max_{s\in[-r,0]}\|\phi(s)\|$, and define $u_t\in C([-r,0],X)$ by $u_t(s)=u(t+s)$, $s\in[-r,0]$, for a function u.

Eq. (1.1) has received some attention recently. For example, Xiang and Ahmed [1] studied the existence of periodic solutions of Eq. (1.1), and Oliveira [2] studied the instability for Eq. (1.1) when $A(t)$ is a constant operator. See Liu [3, 4] for related citations for equations without delay.

In this paper, we will study the relationship between the bounded solutions and the periodic solutions. For this purpose, we define a map P along the solution in such a way that, for $u(\cdot,\phi)$ a solution of Eq. (1.1) with the initial function ϕ,

$$P\phi=u_T(\cdot,\phi), \quad \phi\in C([-r,0],X), \tag{1.2}$$

$$\text{(i.e. } (P\phi)(s)=u_T(s,\phi)=u(T+s,\phi), \ s\in[-r,0],\text{)}$$

and then examine when the map P has a fixed point.

The same issue has been studied thoroughly when X is a finite dimensional space. See, for example, Massera [5], Yoshizawa [6], Burton [7] and Haddock [8], where the

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solutions are proven to be bounded and ultimate bounded, and then compact subsets are constructed using bounded sets and the equicontinuity, and finally an asymptotic fixed point theorem (Horn's) is used to obtain the periodic solutions.

It is possible to carry all the techniques used in finite dimensional spaces to general Banach spaces. However, for a general Banach space, the bounded sets may not be precompact. Thus the construction of compact sets used in finite dimensional spaces will not lead to compact sets in general Banach spaces. Therefore, we need to modify those techniques used in finite dimensional spaces.

One idea is to put some compactness conditions on the resolvent of $A(t)$ so as to prove that the map P is compact. Then compact sets can be obtained using images of the map P on bounded sets.

This idea has been used in, e.g. [1, 9–13] to deal with the same difficulty. It is also used in Liu [3,4] for evolution equations without delay (i.e. $r=0$ in Eq. (1.1)), where it was shown that the equation has a T-periodic solution if solutions of the equation are bounded and ultimate bounded, and that solutions of the equation are bounded and ultimate bounded if there exists a proper Liapunov function.

Recently, after examining the delay equation (1.1) and the methods in Liu [3, 4], we found that the ideas used in [3, 4] for equations without delay can be extended to equations with finite delay. Thus it is the purpose of this paper to report such findings.

Note that the map P maps the initial functions ϕ on $[-r,0]$ to functions $P\phi$ on $[T-r,T]$ along the solutions. If $T-r\leq 0$, then the restrictions of $P\phi$ on $[T-r,0]$ are parts of the initial functions ϕ, and they may be arbitrary, or "bad", i.e. not precompact. To avoid this, we require $T-r>0$ in this paper. Under this assumption, conditions can be given to show that the map P is compact, which enables us to carry the results for equations without delay to the finite delay evolution equation (1.1).

We will obtain the solutions of Eq. (1.1) in Section 2, and study the boundedness and ultimate boundedness of the solutions in Section 3, and then derive the periodic solutions from the boundedness and ultimate boundedness in Section 4.

2. Solving the Eq. (1.1)

In this section we study the existence and uniqueness of solutions for Eq. (1.1). We make the following assumptions.

Assumption 2.1. For a constant $T>r$, $f(t+T,x,y)=f(t,x,y)$, $A(t+T)=A(t)$, $t\geq 0$. f is continuous in its variables and is locally Lipschitzian in the second and the third variables, and maps a bounded set into a bounded set.

Assumption 2.2 ([13, p. 150]). For $t\in[0,T]$,

(H1) The domain $D(A(t))=D$ is independent of t and is dense in X.

(H2) For $t\geq 0$, the resolvent $R(\lambda,A(t))=(\lambda I-A(t))^{-1}$ exists for all λ with $\mathrm{Re}\,\lambda\leq 0$ and is compact, and there is a constant M independent of λ and t such that

$$\|R(\lambda,A(t))\|\leq M(|\lambda|+1)^{-1},\quad \mathrm{Re}\,\lambda\leq 0.$$

(H3) There exist constants L and $0<a\leq 1$ such that

$$\|(A(t)-A(s))A(r)^{-1}\|\leq L|t-s|^{a},\quad s,t,r\in[0,T].$$

Under these assumptions, the results in, e.g. Amann [9], Friedman [10], Kielhöfer [12] and Pazy [13] imply the existence of a unique evolution system $U(t,s)$, $0\leq s\leq t\leq T$, for Eq. (1.1). See [9, 10, 12] and [13] for details.

Theorem 2.1. *Let the Assumptions 2.1–2.2 be satisfied and let $\phi\in C([-r,0],X)$. Then there exists a constant $\alpha>0$ and a unique continuous function $u:[-r,\alpha]\to X$ such that $u_0=\phi$ and*

$$u(t)=U(t,0)\phi(0)+\int_0^t U(t,h)f(h,u(h),u_h)\,\mathrm{d}h,\quad t\in[0,\alpha]. \tag{2.1}$$

Proof. We only need to set up the framework for the use of the Contraction Mapping Theorem. With $\phi\in C([-r,0],X)$ being fixed and with $\alpha>0$ yet to be determined, we define a map Q on $C([-r,\alpha],X)$ such that, for $u\in C([-r,\alpha],X)$ with $u_0=\phi$, $(Qu)(s)=\phi(s)$ for $s\in[-r,0]$, and

$$(Qu)(t)=U(t,0)\phi(0)+\int_0^t U(t,h)f(h,u(h),u_h)\,\mathrm{d}h,\quad \text{for } t\in[0,\alpha]. \tag{2.2}$$

Using the property of the evolution system U, we have $Q:C([-r,\alpha],X)\to C([-r,\alpha],X)$. Next, for $u,v\in C([-r,\alpha],X)$ with $u_0=v_0=\phi$ and $t\in[0,\alpha]$, one has

$$(Qu)(t)-(Qv)(t)=\int_0^t U(t,h)[f(h,u(h),u_h)-f(h,v(h),v_h)]\,\mathrm{d}h \tag{2.3}$$

$$=\int_0^t U(t,h)[f(h,u(h),u_h)-f(h,u(h),v_h)]\,\mathrm{d}h$$

$$+\int_0^t U(t,h)[f(h,u(h),v_h)-f(h,v(h),v_h)]\,\mathrm{d}h. \tag{2.4}$$

Now, f is locally Lipschitzian in the second and the third variables and $U(t,h)$ is a bounded operator, it is then clear that we can obtain the result by using the Contraction Mapping Theorem. Details will be left here. □

Note that the function u determined by Theorem 2.1 is often called "the mild solution of Eq. (1.1)". In our study here, we would like the map P to be defined on the whole space $C([-r,0],X)$, i.e. solutions of Eq. (1.1) exist for all initial functions in $C([-r,0],X)$. So that in this paper, "a solution of Eq. (1.1)" means a "a mild solution", i.e. a function determined by Theorem 2.1. Under some extra conditions on ϕ, $A(t)$ and f, mild solutions give rise to "classical solutions" (i.e. have derivatives). See e.g. [1, 9, 10, 12, 13] for details. Also note that we are concerned with the bounded and periodic solutions here, so we may assume that solutions exist on $[0,\infty)$. We will write $u=u(\cdot,\phi)$ for a solution with the initial function ϕ.

3. Boundedness and ultimate boundedness

Consider the finite delay evolution equation

$$u'(t) + A(t)u(t) = f(t, u(t), u_t), \quad t > 0, \quad u(s) = \phi(s), \ s \in [-r, 0], \tag{3.1}$$

in Banach space $(X, \|\cdot\|)$. The following definitions are standard, we state them here for convenient references. Note that the uniform boundedness and uniform ultimate boundedness are not required to obtain the periodic solutions here, so we only define the boundedness and ultimate boundedness. See [7] for more references.

Definition 3.1. Solutions of Eq. (3.1) are bounded if for each $B_1 > 0$, there is a $B_2 > 0$, such that $\|\phi\|_C \leq B_1$ and $t \geq 0$ imply $\|u(t, \phi)\| < B_2$.

Definition 3.2. Solutions of Eq. (3.1) are ultimate bounded if there is a bound $B > 0$, such that for each $B_3 > 0$, there is a $K > 0$, such that $\|\phi\|_C \leq B_3$ and $t \geq K$ imply $\|u(t, \phi)\| < B$.

Definition 3.3. An operator $P : Z \to Z$ is called compact on Z if P maps bounded sequence (or set) into precompact sequence (or set).

Lemma 3.1 ((Horn's Fixed Point Theorem) [7]). *Let $E_0 \subset E_1 \subset E_2$ be convex subsets of Banach space Z, with E_0 and E_2 compact subsets and E_1 open relative to E_2. Let $P : E_2 \to Z$ be a continuous map such that for some positive integer m,*

$$P^j(E_1) \subset E_2, \quad 1 \leq j \leq m - 1,$$
$$P^j(E_1) \subset E_0, \quad m \leq j \leq 2m - 1.$$

Then P has a fixed point in E_0.

Now, we state the following result which asserts that the existence of a proper Liapunov function implies the boundedness and ultimate boundedness of the solutions. The result is analogous to the one in Burton [7], and a similar result for equations without delay is proved in Liu [4]. Thus we only sketch the proof here.

Theorem 3.1. *Assume that there exist functions ("wedges") W_i, $i = 1, 2, 3$, with $W_i : [0, \infty) \to [0, \infty)$, $W_i(0) = 0$, W_i strictly increasing, and $W_1(t) \to \infty$, $t \to \infty$. Further, assume that there exists a (Liapunov) function $V : X \to \Re$ (reals), such that for some constant $M > 0$, when u is a solution of Eq. (3.1) with $\|u(t)\| \geq M$, then*
(a) $W_1(\|u(t)\|) \leq V(u(t)) \leq W_2(\|u(t)\|)$,
(b) $\mathrm{d}/\mathrm{d}t V(u(t)) \leq -W_3(\|u(t)\|)$, *or* $V(u(t)) - V(u(0)) \leq -\int_0^t W_3(\|u(s)\|)\,\mathrm{d}s$.
Then solutions of equation (3.1) are bounded and ultimate bounded.

Proof. Let $u(t) = u(t, \phi)$ and let $B_1 > 0$ be given with $B_1 \geq M$. Find $B_2 \geq B_1$ with $W_1(B_2) = W_2(B_1)$. If for some interval $[t_1, t_2]$ with $0 \leq t_1$, $\|u(t_1)\| = B_1$, and $\|u(t)\| \geq$

B_1 on $[t_1, t_2]$, then for $t \in [t_1, t_2]$,

$$\begin{aligned} W_1(\|u(t)\|) &\le V(u(t)) \le V(u(t_1)) \\ &\le W_2(\|u(t_1)\|) \le W_2(B_1) = W_1(B_2), \end{aligned} \tag{3.2}$$

and hence $\|u(t)\| \le B_2$. This implies that if $\|\phi\|_C \le B_1$, then $\|u(t)\| \le B_2$, $t \ge 0$, which gives the boundedness.

Next, we prove the ultimate boundedness. Find $B \ge M+1$ with $W_1(B) = W_2(M+1)$ and let $B_3 > 0$ be given. We need to prove that there is a $K > 0$ such that if $\|\phi\|_C \le B_3$ and $t \ge K$, then $\|u(t)\| \le B$.

Now, if $\|u(t)\| > B \ge M+1$ for $t \ge 0$, then

$$0 < W_1(M+1) \le V(u(t)) \le V(u(0)) - W_3(M+1)t \le W_2(B_3) - W_3(M+1)t.$$

This fails when $t \ge W_2(B_3)/W_3(M+1)$. Thus one can verify that

$$K \equiv W_2(B_3)/W_3(M+1)$$

will work. (See [4] for details.) □

Remark 3.1. As can be seen here and in Liu [4], the proof of Theorem 3.1 is independent of the form of equation (3.1). Thus Theorem 3.1 is a general result, and can be applied to many other equations.

4. Periodic solutions

In this section, we study the periodic solutions for the finite delay evolution equation

$$u'(t) + A(t)u(t) = f(t, u(t), u_t), \quad t > 0, \quad u(s) = \phi(s), \; s \in [-r, 0]. \tag{4.1}$$

For this purpose, we define a map P along the solution in such a way that, for $u(\cdot, \phi)$ a solution of Eq. (4.1) with the initial function ϕ,

$$P\phi = u_T(\cdot, \phi), \quad \phi \in C([-r, 0], X), \tag{4.2}$$

and then examine whether the map P has a fixed point. We note that a fixed point of P gives rise to a periodic solution. Because if $P\phi = \phi$, then for the solution $u(\cdot) = u(\cdot, \phi)$ with $u_0(\cdot, \phi) = \phi$, we can define $y(t) = u(t+T)$. Now, for $t \ge 0$, we can use the known formulas ([13]) $U(t,s) = U(t,r)U(r,s)$ and $U(t+T, s+T) = U(t,s)$ (since the operator $A(t)$ is T-periodic in t) to obtain

$$\begin{aligned} y(t) = u(t+T) &= U(t+T, 0)\phi(0) + \int_0^{t+T} U(t+T, h) f(h, u(h), u_h)\,\mathrm{d}h \\ &= U(t+T, T)U(T, 0)\phi(0) + \int_0^T U(t+T, h) f(h, u(h), u_h)\,\mathrm{d}h \\ &\quad + \int_T^{t+T} U(t+T, h) f(h, u(h), u_h)\,\mathrm{d}h \end{aligned}$$

$$= U(t,0)U(T,0)\phi(0) + \int_0^T U(t+T,T)U(T,h)f(h,u(h),u_h)\,\mathrm{d}h$$

$$+ \int_0^t U(t+T,T+s)f(T+s,u(T+s),u_{T+s})\,\mathrm{d}s$$

$$= U(t,0)U(T,0)\phi(0) + \int_0^T U(t,0)U(T,h)f(h,u(h),u_h)\,\mathrm{d}h$$

$$+ \int_0^t U(t,s)f(s,y(s),y_s)\,\mathrm{d}s$$

$$= U(t,0)\left[U(T,0)\phi(0) + \int_0^T U(T,h)f(h,u(h),u_h)\,\mathrm{d}h\right]$$

$$+ \int_0^t U(t,s)f(s,y(s),y_s)\,\mathrm{d}s$$

$$= U(t,0)u(T) + \int_0^t U(t,s)f(s,y(s),y_s)\,\mathrm{d}s$$

$$= U(t,0)y(0) + \int_0^t U(t,s)f(s,y(s),y_s)\,\mathrm{d}s. \tag{4.3}$$

This implies that y is also a solution, and $y_0 = u_T(\phi) = P\phi = \phi$. Then the uniqueness implies that $(u(t+T) =)y(t) = u(t)$, so that $u(\phi)$ is a T-periodic solution.

The following lemma from Amann [9] will be used here to show that P is a compact operator. Recall that in the usual way (see, e.g. Amann [9], Friedman [10], Pazy [13]) we define fractional power operator A^α and Banach space X_α for $0 \le \alpha \le 1$, where $A = A(0)$ and $X_\alpha = (D(A^\alpha),\ \|\cdot\|_\alpha)$ with $\|x\|_\alpha \equiv \|A^\alpha x\|$. We also write the norm in $L(X_\alpha, X_\beta)$ (space of bounded linear operators from X_α to X_β) as $\|\cdot\|_{\alpha,\beta}$.

Lemma 4.1 ([9]). (i) *Suppose that* $0 \le \alpha \le \beta < 1$. *Then for* $\beta - \alpha < \gamma < 1$, *there is a constant* $C(\alpha,\beta,\gamma)$ *such that*

$$\|U(t,h)\|_{\alpha,\beta} \le C(\alpha,\beta,\gamma)(t-h)^{-\gamma}, \quad 0 \le h < t \le T.$$

(ii) *For* $0 \le \gamma < 1$, *there is a constant* $C(\gamma)$, *such that for* $g \in C([0,L],X)$ ($L > 0$ *is a constant*), *one has for* $0 \le s,\ t \le L$,

$$\left\|\int_0^t U(t,h)g(h)\,\mathrm{d}h - \int_0^s U(s,h)g(h)\,\mathrm{d}h\right\| \le C(\gamma)|t-s|^\gamma \max_{0 \le h \le L} \|g(h)\|.$$

(iii) *Let* $0 \le \alpha < \beta \le 1$. *Then*

$$K(x,g)(t) \equiv U(t,0)x + \int_0^t U(t,h)g(h)\,\mathrm{d}h, \quad 0 \le t \le T,$$

defines a continuous linear operator from $X_\beta \times C([0,T],X)$ *into* $C^\gamma([0,T],X_\alpha)$ *for every* $\gamma \in [0, \beta - \alpha)$.

We also state the Arzela Theorem for Banach spaces here for convenient reference.

Lemma 4.2. *Let* $E \subset C([a,b],X)$ *be bounded. Then* E *is precompact if and only if functions in* E *are equicontinuous and for each* $t \in [a,b]$, *the set* $\{f(t) : f \in E\}$ *is precompact in* X.

Now we show that P defined by (4.2) is a compact operator. The idea in the proof is similar to the one in [4].

Theorem 4.1. *Let the Assumptions* 2.1–2.2 *be satisfied, and assume that solutions of Eq.* (4.1) *are bounded. Then* $P : C([-r,0],X) \to C([-r,0],X)$ *defined by* (4.2) *is a compact operator.*

Proof. Let $H \subset C([-r,0],X)$ be bounded. Since solutions of Eq. (4.1) are bounded, it follows that $E = P(H) \subset C([-r,0],X)$ is bounded. In the following, we will use the Arzela Theorem to show that E is precompact.

For $s \in [-r,0]$, a function in E can be expressed as

$$\begin{aligned}(P\phi)(s) &= u_T(s,\phi) = u(T+s,\phi)\\ &= U(T+s,0)\phi(0) + \int_0^{T+s} U(T+s,h) f(h,u(h),u_h)\,\mathrm{d}h, \quad \phi \in H. \end{aligned} \tag{4.4}$$

Since $T - r > 0$, there is $k > 0$ such that $T + s > k$ for $s \in [-r,0]$. From [13, p.164], one has for $s \in [-r,0]$,

$$U(T+s,0)\phi(0) = U(T+s,k)U(k,0)\phi(0), \quad \phi \in H. \tag{4.5}$$

Fix $\eta \in (0,1)$. Then from Lemma 4.1(i), $U(k,0) : X \to X_\eta$ is bounded. Next the embedding $X_\eta \to X$ is compact (under Assumption 2.2(H2), see e.g. [11]), thus $\{U(k,0)\phi(0) : \phi \in H\}$ is precompact in X since $\{\phi(0) : \phi \in H\}$ is bounded in X. Therefore, the closure of $\{U(k,0)\phi(0) : \phi \in H\}$ is compact in X. Now, one can verify that as functions on $\cdot \in [-r,0]$,

$$\{U(T+\cdot,0)\phi(0) : \phi \in H\} = \{U(T+\cdot,k)[U(k,0)\phi(0)] : \phi \in H\} \tag{4.6}$$

is equicontinuous. Next, from Lemma 4.1(ii), for $0 \le \gamma < 1$, there is a constant $C(\gamma)$, such that for $s_1, s_2 \in [-r,0]$,

$$\left\| \int_0^{T+s_2} U(T+s_2,h) f(h,u(h),u_h)\,\mathrm{d}h - \int_0^{T+s_1} U(T+s_1,h) f(h,u(h),u_h)\,\mathrm{d}h \right\| \tag{4.7}$$

$$\le C(\gamma)|s_1 - s_2|^\gamma \max_{0 \le h \le T} \|f(h,u(h),u_h)\|. \tag{4.8}$$

Since solutions of Eq. (4.1) are bounded and f maps a bounded set into a bounded set, there exists $M_1 = M_1(H) > 0$ such that

$$\|f(t,u(t,\phi),u_t(\phi))\| \le M_1, \quad t\in[0,T], \ \phi\in H.$$

Thus, as functions on $\cdot \in [-r,0]$,

$$\left\{\int_0^{T+\cdot} U(T+\cdot,h)f(h,u(h),u_h)\,\mathrm{d}h : \phi\in H\right\} \tag{4.9}$$

is also equicontinuous. Therefore, functions in E are equicontinuous.

Next, fix $s_0\in[-r,0]$. From the above arguments, we also know that

$$\{U(T+s_0,0)\phi(0) : \phi\in H\} \tag{4.10}$$

is precompact in X. Now for $\phi\in H$, we may let $g(t)=f(t,u(t,\phi),u_t(\phi))$. Then

$$\int_0^{T+s_0} U(T+s_0,h)f(h,u(h,\phi),u_h(\phi))\,\mathrm{d}h \equiv K(0,g)(T+s_0)\in X_\eta$$

according to Lemma 4.1(iii). Also note that by Lemma 4.1(i), there are constants $\gamma\in(0,1)$ and $M_2>0$ such that

$$\|U(T+s_0,h)\|_{0,\eta} \le M_2(T+s_0-h)^{-\gamma}, \quad 0\le h<T+s_0.$$

Thus

$$\left\|\int_0^{T+s_0} U(T+s_0,h)f(h,u(h,\phi),u_h(\phi))\,\mathrm{d}h\right\|_\eta \le M_1M_2T^{1-\gamma}/(1-\gamma), \quad \phi\in H.$$

Therefore

$$\left\{\int_0^{T+s_0} U(T+s_0,h)f(h,u(h,\phi),u_h(\phi))\,\mathrm{d}h : \phi\in H\right\} \tag{4.11}$$

is bounded in X_η. Then use the fact that the embedding $X_\eta\to X$ is compact again, we see that the set defined by (4.11) is precompact in X. Now the Arzela Theorem implies that the map P is a compact operator. □

Next we show that periodic solutions can be derived from the boundedness and ultimate boundedness of solutions. The proof is analogous to the one in [4]. Thus we only sketch the proof here.

Theorem 4.2. *Let the Assumptions* 2.1–2.2 *be satisfied. If the solutions of Eq.* (4.1) *are bounded and ultimate bounded, then Eq.* (4.1) *has a T-periodic solution.*

Proof. Let the map P be defined by (4.2). Using an argument similar to (4.3), we see that

$$P^m(\phi)=u_{mT}(\phi),\quad \phi\in C([-r,0]). \tag{4.12}$$

Next, let $B>0$ be the bound in the definition of ultimate boundedness. Using boundedness, there is $B_2>B$ such that $\{\|\phi\|_C\le B,\ t\ge 0\}$ implies $\|u(t,\phi)\|<B_2$. And also, there is $B_4>2B_2$ such that $\{\|\phi\|_C\le 2B_2,\ t\ge 0\}$ implies $\|u(t,\phi)\|<B_4$. Next, using ultimate boundedness, there is a positive integer m such that $\{\|\phi\|_C\le 2B_2, t\ge (m-2)T\}$ implies $\|u(t,\phi)\|<B$. These imply

$$\begin{aligned}\|P^{i-1}\phi\|_C&=\|u((i-1)T+\cdot,\phi)\|_C<B_4\\ &\qquad\text{for } i=1,2,3,\dots \text{ and } \|\phi\|_C\le 2B_2,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\|P^{i-1}\phi\|_C&=\|u((i-1)T+\cdot,\phi)\|_C<B\\ &\qquad\text{for } i\ge m \text{ and } \|\phi\|_C\le 2B_2.\end{aligned} \tag{4.14}$$

Now let

$$\begin{aligned}H&\equiv\{\phi\in C([-r,0],X):\|\phi\|_C<B_4\},\quad E_2\equiv \mathrm{cl.(cov.}\ (P(H))),\\ K&\equiv\{\phi\in C([-r,0],X):\|\phi\|_C<2B_2\},\quad E_1\equiv K\cap E_2,\\ G&\equiv\{\phi\in C([-r,0],X):\|\phi\|_C<B\},\quad E_0\equiv \mathrm{cl.(cov.}\ (P(G))),\end{aligned} \tag{4.15}$$

where cov. (F) is the convex hull of the set F defined by cov. $(F)=\{\sum_{i=1}^n \lambda_i f_i: n\ge 1, f_i\in F,\ \lambda_i\ge 0,\ \sum_{i=1}^n\lambda_i=1\}$, and cl. denotes the closure. Thus E_0, E_1 and E_2 are convex subsets and E_1 is open relative to E_2.

Then one can show that $E_0\subset E_1\subset E_2$ and that E_0 and E_2 are compact sets. And from (4.13) and (4.14), one has

$$P^i(E_1)\subset P^i(K)=PP^{i-1}(K)\subset P(H)\subset E_2,\quad i=1,2,3,\dots, \tag{4.16}$$

$$P^i(E_1)\subset P^i(K)=PP^{i-1}(K)\subset P(G)\subset E_0,\quad i\ge m. \tag{4.17}$$

Finally, we verify the continuity of the map P. Let $u(t)=u(t,\phi_1)$ and $v(t)=v(t,\phi_2)$. Then for $t\in[0,T]$,

$$u(t)=U(t,0)\phi_1(0)+\int_0^t U(t,h)f(h,u(h),u_h)\,\mathrm{d}h, \tag{4.18}$$

$$v(t)=U(t,0)\phi_2(0)+\int_0^t U(t,h)f(h,v(h),v_h)\,\mathrm{d}h. \tag{4.19}$$

By using the Lipschitzian condition, we can find constants $C_i>1$, $i=1,2,\dots,$ such that for $t\in[0,T]$,

$$\begin{aligned}\|u(t)-v(t)\|\le\|&U(t,0)[\phi_1(0)-\phi_2(0)]\\ &+\int_0^t U(t,h)[f(h,u(h),u_h)-f(h,u(h),v_h)]\,\mathrm{d}h\end{aligned}$$

$$+\int_0^t U(t,h)[f(h,u(h),v_h)-f(h,v(h),v_h)]\,\mathrm{d}h\| \tag{4.20}$$

$$\le C_1\|\phi_1(0)-\phi_2(0)\|+\int_0^t C_2\|u_h-v_h\|_C\,\mathrm{d}h$$

$$+\int_0^t C_3\|u(h)-v(h)\|\,\mathrm{d}h \tag{4.21}$$

$$\le C_1\|\phi_1-\phi_2\|_C+\int_0^t C_4\|u_h-v_h\|_C\,\mathrm{d}h. \tag{4.22}$$

Then for $0\le s\le t\le T$,

$$\|u(s)-v(s)\|\le C_1\|\phi_1-\phi_2\|_C+\int_0^s C_4\|u_h-v_h\|_C\,\mathrm{d}h$$

$$\le C_1\|\phi_1-\phi_2\|_C+\int_0^t C_4\|u_h-v_h\|_C\,\mathrm{d}h. \tag{4.23}$$

Therefore,

$$\|u_t-v_t\|_C\le C_1\|\phi_1-\phi_2\|_C+\int_0^t C_4\|u_h-v_h\|_C\,\mathrm{d}h,\quad t\in[0,T]. \tag{4.24}$$

Hence the Gronwall's inequality can be applied to get

$$\|u_t-v_t\|_C\le C_5\|\phi_1-\phi_2\|_C,\quad t\in[0,T]. \tag{4.25}$$

Thus the map P is continuous. Now, it is clear that the Horn's Fixed Point Theorem can be used to get a T-periodic solution of Eq. (4.1). □

Remark 4.1. As can be seen here and in [4], the proof of Theorem 4.2 is independent of the form of Eq. (4.1). Thus it can be applied to other equations as well. Combine this with Remark 3.1, we have the following general results for differential equations, with or without delays.

(A) For a differential equation in a general Banach space, if there exists a Liapunov function that satisfies the conditions (a) and (b) in Theorem 3.1, then the solutions of the equation are bounded and ultimate bounded.

(B) For a differential equation in a general Banach space, if the following conditions are satisfied,

1. The solutions of the equation are bounded and ultimate bounded,
2. the equation is T-periodic in time t,
3. the map that maps an initial function (could be just a point if without delay) along the solution by T units is compact,

then the equation has a T-periodic solution.

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