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Hyperbolic singular perturbations for integrodifferential equations

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Abstract

We study the convergence of solutions of

$$\begin{cases} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon) + \int_0^t K(t-s)(\varepsilon^2 A + B)u(s;\varepsilon)\,\mathrm{d}s \\ \qquad\qquad +f(t;\varepsilon), \quad t \geqslant 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \quad u'(0;\varepsilon) = u_1(\varepsilon), \end{cases}$$

to solutions of

$$\begin{cases} w'(t) = Bw(t) + \int_0^t K(t-s)Bw(s)\,\mathrm{d}s + f(t), \quad t \geqslant 0, \\ w(0) = w_0, \end{cases}$$

when $\varepsilon \to 0$. Here A and B are linear unbounded operators in a Banach space X, $K(t)$ is a linear bounded operator for each $t \geqslant 0$ in X, and $f(t;\varepsilon)$ and $f(t)$ are given X-valued functions. Our result extends the studies in Fattorini [J. Diff. Eq. 70 (1987) 1] for equations without the integral term and in Liu [Proc. Am. Math. Soc. 122 (1994) 791] for parabolic singular perturbation problems.

Keywords: Hyperbolic singular perturbations; Convergence in solutions; Integrodifferential equations in Banach spaces

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1. Introduction

This paper is concerned with hyperbolic singular perturbation problems for integrodifferential equations. For references in this area, we refer the reader to the monographs of Fattorini [4], Goldstein [6] and Smith [11], and the papers of Fattorini [5], Grimmer and Liu [7], Hale and Raugel [9], and the references therein.

In Fattorini [5], the following hyperbolic partial differential equation with a small positive parameter ε from traffic flow

$$\varepsilon^2\left(\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial u}{\partial t} - b\frac{\partial u}{\partial x}\right) = 0,$$

is formulated as an abstract differential equation

$$\varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon), \quad t \geqslant 0$$

in a Banach space, with A and B linear unbounded operators satisfying certain conditions. Then the inhomogeneous singular perturbation problem

$$\begin{cases} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon) + f(t;\varepsilon), & t \geqslant 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \quad u'(0;\varepsilon) = u_1(\varepsilon), \end{cases} \tag{1.1}$$

is studied, and it is shown with some conditions on A and B that, as $\varepsilon \to 0$, if $u_0(\varepsilon) \to w_0$, $\varepsilon^2 u_1(\varepsilon) \to 0$, and $f(\cdot;\varepsilon) \to f(\cdot)$, then $u(t;\varepsilon) \to w(t)$ uniformly on compact subsets of $t \geqslant 0$, where $u(t;\varepsilon)$ is the solution of the Cauchy problem (1.1) and w is the solution of the Cauchy problem

$$\begin{cases} w'(t) = Bw(t) + f(t), & t \geqslant 0, \\ w(0) = w_0. \end{cases} \tag{1.2}$$

This is an extension of an earlier result in Fattorini [4] about the parabolic singular perturbation problem

$$\begin{cases} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = Au(t;\varepsilon) + f(t;\varepsilon), & t \geqslant 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \quad u'(0;\varepsilon) = u_1(\varepsilon), \end{cases}$$

and

$$\begin{cases} w'(t) = Aw(t) + f(t), & t \geqslant 0, \\ w(0) = w_0, \end{cases}$$

where the same result mentioned above holds.

Stimulated by the work of Fattorini [5], we study in this paper the convergence of solutions of the following Cauchy problem for a hyperbolic integrodifferential equation

$$\begin{cases} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon) + \int_0^t K(t-s)(\varepsilon^2 A + B)u(s;\varepsilon)\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad\qquad + f(t;\varepsilon), \quad t \geqslant 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \quad u'(0;\varepsilon) = u_1(\varepsilon), \end{cases} \tag{1.3}$$

to solutions of the Cauchy problem

$$\begin{cases} w'(t) = Bw(t) + \int_0^t K(t-s)Bw(s)\,\mathrm{d}s + f(t), \quad t \geqslant 0, \\ w(0) = w_0, \end{cases} \tag{1.4}$$

when $\varepsilon \to 0$. Here A, B and $K(t)$ (for each $t \geqslant 0$) are operators in a Banach space X and satisfy some assumptions (see Section 2 below), and $f(\cdot;\varepsilon)$, $f(\cdot) \in L^1_{\mathrm{loc}}([0,\infty);X)$. Eq. (1.3) models physical problems, such as viscoelasticity. Following Fattorini [5], we call this study as the hyperbolic singular perturbations for integrodifferential equations.

We will show that, as $\varepsilon \to 0$, if $u_0(\varepsilon) \to w_0, \varepsilon^2 u_1(\varepsilon) \to 0$, and $f(\cdot;\varepsilon) \to f(\cdot)$, then $u(t;\varepsilon) \to w(t)$ uniformly on compact subsets of $t \geqslant 0$ for the solution $u(t;\varepsilon)$ of (1.3) and the solution $w(t)$ of (1.4). When $K(\cdot) \equiv 0$, this result goes back to the corresponding results (Theorems 3.3 and 8.4) in Fattorini [5] for equations without the integral term. Also it covers Theorem 2.1 in Liu [10]. See Remark 3.4 below for details.

2. Preliminaries

Throughout this paper, $\varepsilon > 0$, X is a Banach space; $\mathbf{L}(X)$ denotes the space of all continuous linear operators from X to itself; and $D(A)$ stands for the domain of an operator A.

Here we list the basic assumptions and results of Fattorini [5] that will be used in this work. See [5] for details.

Assume that the domain $D(\varepsilon^2 A + B) = D(A) \cap D(B)$ is dense in X; that the homogeneous version of (1.1) ($f(\cdot;\varepsilon) = 0$) has a solution for $u_0(\varepsilon), u_1(\varepsilon)$ in a dense subspace D of X; and that the solutions of the homogeneous version of (1.1) depend continuously on their initial data uniformly on compacts of $t \geqslant 0$. This is equivalent to the following assumption (cf., [4,5]; see also [14,15]).

(A1) $\varepsilon^2 A + B$ is the generator of a strongly continuous cosine function on X.

Under this condition, one can define two propagators of the homogeneous version of (1.1) by

$$Q(t;\varepsilon)u = u(t;\varepsilon), \quad G(t;\varepsilon)u = v(t;\varepsilon), \quad u \in D, \ t \geqslant 0,$$

where $u(t;\varepsilon)$ (resp. $v(t;\varepsilon)$) is the solution of the homogeneous version of (1.1) with $u(0;\varepsilon) = u$, $u'(0;\varepsilon) = 0$ (resp. with $v(0;\varepsilon) = 0$, $v'(0;\varepsilon) = \varepsilon^{-2}u$); these propagators can be extended to all of X as bounded operators, which we denote by the same symbol; and these operator-valued functions are strongly continuous in $t \geqslant 0$. It is also shown in [5] that the solutions of (1.1) are given by

$$u(t;\varepsilon) = Q(t;\varepsilon)u_0(\varepsilon) + G(t;\varepsilon)[\varepsilon^2 u_1(\varepsilon)] + \int_0^t G(t-s;\varepsilon)f(s;\varepsilon)\,\mathrm{d}s, \tag{2.1}$$

and that for $u \in X$

$$\varepsilon^2 G'(t;\varepsilon)u = Q(t;\varepsilon)u - G(t;\varepsilon)u. \tag{2.2}$$

Following Fattorini [5], we also make the following assumptions.

(A2) There exist constants C, ω, ε_0 independent of t and ε such that for $t \geqslant 0$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0$

$$\|Q(t;\varepsilon)\|, \|G(t;\varepsilon)\| \leqslant C\mathrm{e}^{\omega t}. \tag{2.3}$$

(A3) The restriction B_0 of B to $D(A)$ is closable and there is a v such that $(\lambda - B_0)D(B_0)$ is dense in X for $\operatorname{Re}\lambda > v$.

By virtue of [5, Theorems 3.2 and 8.3] we know that under these assumptions, the closure $\overline{B_0}$ of B_0 generates a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ satisfying

$$\|S(t)\| \leqslant M\mathrm{e}^{\mu t}, \quad t \geqslant 0 \tag{2.4}$$

for certain constants M and μ; and

$$\lim_{\varepsilon\to 0} Q(t,\varepsilon)u = S(t)u, \quad u \in X, \tag{2.5}$$

$$\lim_{\varepsilon\to 0} \left[G(t,\varepsilon) + \mathrm{e}^{-\frac{t}{\varepsilon^2}}\right]u = S(t)u, \quad u \in X, \tag{2.6}$$

uniformly on compact subset of $t \geqslant 0$.

To link the semigroup $\{S(t)\}_{t\geqslant 0}$ and Eq. (1.2), we assume

(A4) $\overline{B_0} = B$.

Therefore, under the assumption (A4), the solutions of (1.2) are given by

$$w(t) = S(t)w_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad w_0 \in D(\overline{B_0}). \tag{2.7}$$

The following is an assumption made specially for the integrodifferential equations (1.3) and (1.4).

(A5) $\{K(t)\}_{t\geqslant 0} \subset \mathbf{L}(X)$. For each $x \in X$, $K(\cdot)x \in W^{2,1}_{\mathrm{loc}}([0,\infty);X)$. $\|K''(\cdot)\|$ is locally bounded on $[0,\infty)$. Here K'' is the strong derivative.

Definition 2.1. An X-valued function $u(\cdot;\varepsilon)$ on $[0,\infty)$ is called a solution of (1.3) if $u(\cdot;\varepsilon)$ is twice continuously differentiable, $u(t;\varepsilon) \in D(A) \cap D(B)$ for $t \geqslant 0$

and (1.3) is satisfied. Similarly, an X-valued function $w(\cdot)$ on $[0,\infty)$ is called a solution of (1.4) if $w(\cdot)$ is continuously differentiable, $w(t) \in D(B)$ for $t \geqslant 0$ and (1.4) is satisfied.

Suppose that $u(t;\varepsilon)$ is a solution of (1.3). As in [4,5,10], we write

$$u(t;\varepsilon) = \mathrm{e}^{-\frac{t}{\varepsilon^2}} v\left(\frac{t}{\varepsilon}\right).$$

Then by (1.3) we have

$$\begin{cases} v''\left(\frac{t}{\varepsilon}\right) = \left(\varepsilon^2 A + B + \frac{1}{4\varepsilon^2}\right) v\left(\frac{t}{\varepsilon}\right) + \int_0^t K(t-s)\mathrm{e}^{\frac{t-s}{2\varepsilon^2}}(\varepsilon^2 A + B) v\left(\frac{s}{\varepsilon}\right) \mathrm{d}s + \mathrm{e}^{\frac{t}{2\varepsilon^2}} f(t;\varepsilon) \\ v(0;\varepsilon) = u_0(\varepsilon), \quad v'(0;\varepsilon) = \frac{1}{2\varepsilon} u_0(\varepsilon) + \varepsilon u_1(\varepsilon), \end{cases}$$

that is

$$\begin{cases} v''(t) = \left(\varepsilon^2 A + B + \frac{1}{4\varepsilon^2}\right) v(t) + \int_0^t \tilde{K}(t-s;\varepsilon)(\varepsilon^2 A + B) v(s) \, \mathrm{d}s + \tilde{f}(t;\varepsilon), \\ v(0;\varepsilon) = u_0(\varepsilon), \quad v'(0;\varepsilon) = \frac{1}{2\varepsilon} u_0(\varepsilon) + \varepsilon u_1(\varepsilon), \end{cases} \tag{2.8}$$

where

$$\tilde{K}(t;\varepsilon) = \varepsilon K(\varepsilon t)\mathrm{e}^{\frac{t}{2\varepsilon}}, \quad \tilde{f}(t;\varepsilon) = f(\varepsilon t;\varepsilon)\mathrm{e}^{\frac{t}{2\varepsilon}}, \quad t \geqslant 0.$$

From [3,12,13], one can find the existence and uniqueness theorems for solutions of (2.8) and (1.4). Since the singular perturbations is what we are concerned in this paper, we assume that (1.3) (i.e., (2.8)) for every $\varepsilon > 0$ and (1.4) have unique solutions, respectively.

3. Singular perturbation theorem

Now we state and prove our main result of the paper concerning singular perturbations for Eqs. (1.3) and (1.4).

Theorem 3.1. *Let $T > 0$ be fixed,* (A1)–(A5) *hold, and*

(A6) $u_0(\varepsilon) \to w_0$, $\varepsilon^2 u_1(\varepsilon) \to 0$, *as* $\varepsilon \to 0$,
(A7) $f(\cdot;\varepsilon) \to f(\cdot)$ *in* $L^1([0,T];X)$, *as* $\varepsilon \to 0$.

Let $u(t;\varepsilon)$ and $w(t)$ be the solution of (1.3) *and* (1.4) *on* $[0,T]$, *respectively. Then*

$$u(t;\varepsilon) \to w(t) \text{ uniformly for } t \in [0,T] \text{ as } \varepsilon \to 0.$$

Proof. Using (A5) and a standard fixed point argument, one can deduce that there exists an $\mathbf{L}(X)$-valued function $F(\cdot)$ such that

$$F(t) + K(t) + \int_0^t K(t-s)F(s)\,\mathrm{d}s = 0,$$

$$F(\cdot)x \in W^{2,1}_{\mathrm{loc}}([0,\infty);X) \quad \text{for each } x \in X,$$

$$\|F'(\cdot)\| \quad \text{is locally bounded on } [0,\infty),$$

where F'' is the strong derivative (cf., [1,2]).

Let $\delta(\cdot)$ be the Dirac measure. Then

$$(\delta + F) * (\delta + K) = \delta. \tag{3.1}$$

Since $u(t;\varepsilon)$ satisfies (1.3), we get

$$\varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\delta + K) * (\varepsilon^2 A + B)u(t;\varepsilon) + f(t;\varepsilon),$$

then by (3.1), we obtain

$$(\delta + F) * [\varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon)] = (\varepsilon^2 A + B)u(t;\varepsilon) + (\delta + F) * f(t;\varepsilon).$$

This means that $u(t;\varepsilon)$ satisfies

$$\begin{cases} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon) + \hat{f}(t;\varepsilon), \\ u(0;\varepsilon) = u_0(\varepsilon), \quad u'(0;\varepsilon) = u_1(\varepsilon), \end{cases} \tag{3.2}$$

where

$$\hat{f}(t;\varepsilon) = (\delta + F) * f(t;\varepsilon) - F * [\varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon)]. \tag{3.3}$$

Similarly, we have

$$\begin{cases} w'(t) = Bw(t) + \hat{f}(t), \quad t \geqslant 0, \\ w(0) = w_0, \end{cases} \tag{3.4}$$

where

$$\hat{f}(t) = (\delta + F) * f(t) - F * w'(t). \tag{3.5}$$

Therefore, by (2.1) and (2.7), we know that

$$u(t;\varepsilon) = Q(t;\varepsilon)u_0(\varepsilon) + G(t;\varepsilon)[\varepsilon^2 u_1(\varepsilon)] + \int_0^t G(t-s;\varepsilon)\hat{f}(s;\varepsilon)\,\mathrm{d}s, \tag{3.6}$$

$$w(t) = S(t)w_0 + \int_0^t S(t-s)\hat{f}(s)\,\mathrm{d}s. \tag{3.7}$$

Thus

$$
\begin{aligned}
u(t;\varepsilon)-w(t) &= Q(t;\varepsilon)u_0(\varepsilon)-Q(t;\varepsilon)w_0+Q(t;\varepsilon)w_0-S(t)w_0\\
&\quad + G(t;\varepsilon)[\varepsilon^2u_1(\varepsilon)]+\int_0^t[G(t-s;\varepsilon)+\mathrm{e}^{-\frac{t-s}{\varepsilon^2}}-S(t-s)]\hat{f}(s)\,\mathrm{d}s\\
&\quad -\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^2}}\hat{f}(s)\,\mathrm{d}s+\int_0^t G(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)]\,\mathrm{d}s.
\end{aligned}
\tag{3.8}
$$

Clearly, (2.3), (A6) and (2.5) imply that, for $t\in[0,T]$ uniformly

$$Q(t;\varepsilon)u_0(\varepsilon)-Q(t;\varepsilon)w_0\to 0,\quad \text{as }\ \varepsilon\to 0, \tag{3.9}$$

$$Q(t;\varepsilon)w_0-S(t)w_0\to 0,\quad \text{as }\varepsilon\to 0, \tag{3.10}$$

$$G(t;\varepsilon)[\varepsilon^2u_1(\varepsilon)]\to 0,\quad \text{as }\ \varepsilon\to 0. \tag{3.11}$$

By (2.3), (2.4), (2.6), (3.5) and the properties of $F(\cdot)$, we have, for $t\in[0,T]$ uniformly

$$\int_0^t\Big[G(t-s;\varepsilon)+\mathrm{e}^{-\frac{t-s}{\varepsilon^2}}-S(t-s)\Big]\hat{f}(s)\,\mathrm{d}s\to 0,\quad \text{as }\ \varepsilon\to 0. \tag{3.12}$$

Since for each $t>0$,

$$\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^2}}\,\mathrm{d}s=\varepsilon^2\Big[1-\mathrm{e}^{-\frac{t}{\varepsilon^2}}\Big]\leqslant\varepsilon^2\to 0,\quad \text{as }\ \varepsilon\to 0, \tag{3.13}$$

we see that for $t\in[0,T]$ uniformly

$$\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^2}}\hat{f}(s)\,\mathrm{d}s\to 0,\quad \text{as }\ \varepsilon\to 0. \tag{3.14}$$

Moreover, from (3.3) and (3.5) and the properties of $F(\cdot)$, we deduce that

$$
\begin{aligned}
\hat{f}(t;\varepsilon)-\hat{f}(t) &= (\delta+F)*[f(t;\varepsilon)-f(t)]-\int_0^t F(t-s)[u(s;\varepsilon)-w(s)]'\,\mathrm{d}s\\
&\quad -\varepsilon^2\int_0^t F(t-s)u''(s;\varepsilon)\,\mathrm{d}s\\
&= (\delta+F)*[f(t;\varepsilon)-f(t)]-F(0)[u(t;\varepsilon)-w(t)]\\
&\quad +F(t)[u_0(\varepsilon)-w_0]+\varepsilon^2F'(t)[u_0(\varepsilon)-w_0]\\
&\quad +\varepsilon^2F'(t)w_0+\varepsilon^2F(t)u_1(\varepsilon)-\int_0^t F'(t-s)[u(s;\varepsilon)-w(s)]\,\mathrm{d}s\\
&\quad -\varepsilon^2\Big[\int_0^t F''(t-s)u(s;\varepsilon)\,\mathrm{d}s+F'(0)u(t;\varepsilon)+F(0)u'(t;\varepsilon)\Big].
\end{aligned}
\tag{3.15}
$$

By (2.3), the properties of $F(\cdot)$, (A6), and (A7), we get

$$\int_0^t \|G(t-s;\varepsilon)F(0)[u(s;\varepsilon)-w(s)]\|\,\mathrm{d}s \leqslant (\mathrm{const})\int_0^t \|u(s;\varepsilon)-w(s)\|\,\mathrm{d}s, \tag{3.16}$$

$$\int_0^t \|G(t-s;\varepsilon)\int_0^s F'(s-r)[u(r;\varepsilon)-w(r)]\,\mathrm{d}r\|\,\mathrm{d}s \\ \leqslant (\mathrm{const})\int_0^t \|u(s;\varepsilon)-w(s)\|\,\mathrm{d}s, \tag{3.17}$$

where "const" means a constant that is independent of ε and $t \in [0,T]$. Also, for $t \in [0,T]$ uniformly

$$\int_0^t G(t-s;\varepsilon)(\delta+F)*[f(s;\varepsilon)-f(s)]\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.18}$$

$$\int_0^t G(t-s;\varepsilon)F(s)[u_0(\varepsilon)-w_0]\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.19}$$

$$\int_0^t G(t-s;\varepsilon)\varepsilon^2 F'(s)[u_0(\varepsilon)-w_0]\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.20}$$

$$\int_0^t G(t-s;\varepsilon)\varepsilon^2 F'(s)w_0\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.21}$$

$$\int_0^t G(t-s;\varepsilon)F(s)[\varepsilon^2 u_1(\varepsilon)]\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.22}$$

Combining (3.8)–(3.12), (3.14)–(3.22), we have

$$\|u(t;\varepsilon)-w(t)\| \leqslant 0(\varepsilon,[0,T]) + (\mathrm{const})\int_0^t \|u(s;\varepsilon)-w(s)\|\,\mathrm{d}s \\ + \left\|\int_0^t G(t-s;\varepsilon)\varepsilon^2\left[\int_0^s F''(s-r)u(r;\varepsilon)\,\mathrm{d}r + F'(0)u(s;\varepsilon) \right.\right. \\ \left.\left. + F(0)u'(s;\varepsilon)\right]\mathrm{d}s\right\|, \tag{3.23}$$

where $0(\varepsilon,[0,T])$ satisfies

$$0(\varepsilon,[0,T]) \to 0 \text{ as } \varepsilon \to 0, \quad \text{uniformly for } t \in [0,T]. \tag{3.24}$$

Next

$$\begin{aligned}
&\varepsilon^2\left[\int_0^s F''(s-r)u(r;\varepsilon)\,\mathrm{d}r + F'(0)u(s;\varepsilon) + F(0)u'(s;\varepsilon)\right] \\
&\quad = \varepsilon^2\left[\int_0^s F''(s-r)u(r;\varepsilon)\,\mathrm{d}r - \int_0^s F''(s-r)w(r)\,\mathrm{d}r + \int_0^s F''(s-r)w(r)\,\mathrm{d}r\right. \\
&\qquad \left. + F'(0)u(s;\varepsilon) - F'(0)w(s) + F'(0)w(s) + F(0)u'(s;\varepsilon)\right] \\
&\quad = \varepsilon^2\int_0^s F''(s-r)[u(r;\varepsilon) - w(r)]\,\mathrm{d}r + \varepsilon^2 F'(0)[u(s;\varepsilon) - w(s)] \\
&\qquad + \varepsilon^2\int_0^s F''(s-r)w(r)\,\mathrm{d}r + \varepsilon^2 F'(0)w(s) + \varepsilon^2 F(0)u'(s;\varepsilon),
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
&\int_0^t G(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)\,\mathrm{d}s \\
&\quad = G(0;\varepsilon)\varepsilon^2 F(0)u(t;\varepsilon) - G(t;\varepsilon)\varepsilon^2 F(0)u_0(\varepsilon) \\
&\qquad + \int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)u(s;\varepsilon)\,\mathrm{d}s \\
&\quad = G(0;\varepsilon)\varepsilon^2 F(0)[u(t;\varepsilon) - w(t)] + G(0;\varepsilon)\varepsilon^2 F(0)w(t) \\
&\qquad - G(t;\varepsilon)\varepsilon^2 F(0)[u_0(\varepsilon) - w_0] - G(t;\varepsilon)\varepsilon^2 F(0)w_0 \\
&\qquad + \int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)[u(s;\varepsilon) - w(s)]\,\mathrm{d}s \\
&\qquad + \int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)w(s)\,\mathrm{d}s.
\end{aligned} \tag{3.26}$$

By (2.3), the properties of $F(\cdot)$, (A6), and the boundedness of $w(\cdot)$ on $[0,T]$, we see that

$$\begin{aligned}
&\left\|\int_0^t G(t-s;\varepsilon)\varepsilon^2\int_0^s F''(s-r)[u(r;\varepsilon) - w(r)]\,\mathrm{d}r\,\mathrm{d}s\right\| \\
&\quad \leqslant (\mathrm{const})\int_0^t \|u(s;\varepsilon) - w(s)\|\,\mathrm{d}s,
\end{aligned} \tag{3.27}$$

$$\left\|\int_0^t G(t-s;\varepsilon)\varepsilon^2 F'(0)[u(s;\varepsilon) - w(s)]\,\mathrm{d}s\right\| \leqslant (\mathrm{const})\int_0^t \|u(s;\varepsilon) - w(s)\|\,\mathrm{d}s, \tag{3.28}$$

$$\left\|\int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)[u(s;\varepsilon) - w(s)]\,\mathrm{d}s\right\| \leqslant (\mathrm{const})\int_0^t \|u(s;\varepsilon) - w(s)\|\,\mathrm{d}s, \tag{3.29}$$

$$\|G(0;\varepsilon)\varepsilon^2F(0)\| \leqslant \frac{1}{2}, \quad \text{for } \varepsilon \text{ small enough,} \tag{3.30}$$

and for $t \in [0,T]$ uniformly

$$\int_0^t G(t-s;\varepsilon)\varepsilon^2 \int_0^s F''(s-r)w(r)\,\mathrm{d}r\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.31}$$

$$\int_0^t G(t-s;\varepsilon)\varepsilon^2F'(0)w(s)\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.32}$$

$$G(0;\varepsilon)\varepsilon^2F(0)w(t) \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.33}$$

$$G(t;\varepsilon)\varepsilon^2F(0)[u_0(\varepsilon)-w_0] \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.34}$$

$$G(t;\varepsilon)\varepsilon^2F(0)w_0 \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.35}$$

Moreover, it follows from (2.2) that

$$\begin{aligned}
&\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)w(s)\,\mathrm{d}s \\
&\quad = \int_0^t [Q(t-s;\varepsilon)-G(t-s;\varepsilon)]F(0)w(s)\,\mathrm{d}s \\
&\quad = \int_0^t [Q(t-s;\varepsilon)-S(t-s)]F(0)w(s)\,\mathrm{d}s \\
&\qquad + \int_0^t [S(t-s)-\mathrm{e}^{-\frac{t-s}{\varepsilon^2}}-G(t-s;\varepsilon)]F(0)w(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^2}}F(0)w(s)\,\mathrm{d}s.
\end{aligned} \tag{3.36}$$

By (2.3)–(2.6) and (3.13), we get, for $t \in [0,T]$ uniformly

$$\int_0^t [Q(t-s;\varepsilon)-S(t-s)]F(0)w(s)\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.37}$$

$$\int_0^t \left[S(t-s)-\mathrm{e}^{-\frac{t-s}{\varepsilon^2}}-G(t-s;\varepsilon)\right]F(0)w(s)\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.38}$$

$$\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon^2}}F(0)w(s)\,\mathrm{d}s \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.39}$$

Consequently, a combination of (3.23), (3.25)–(3.39) shows that

$$\|u(t;\varepsilon)-w(t)\| \leqslant 0(\varepsilon,[0,T]) + (\text{const})\int_0^t \|u(s;\varepsilon)-w(s)\|\,\mathrm{d}s, \quad t \in [0,T].$$

This, together with Gronwall's inequality (cf., e.g., [8]), implies that

$$\|u(t;\varepsilon)-w(t)\| \leqslant 0(\varepsilon,[0,T]), \quad t \in [0,T].$$

This completes the proof. □

Theorem 3.2. *Let $T > 0$ be fixed, and* (A1), (A2), (A5)–(A7) *hold. Also, assume that B generates a strongly continuous semigroup on X and $D(A) \cap D(B)$ is a core of B. Let $u(t;\varepsilon)$ and $w(t)$ be the solution of* (1.3) *and* (1.4) *on* $[0, T]$, *respectively. Then*

$$u(t;\varepsilon) \to w(t) \text{ uniformly for } t \in [0, T] \text{ as } \varepsilon \to 0.$$

Proof. Since B generates a strongly continuous semigroup on X, and $D(A) \cap D(B)$ is a core of B, we see that (A3) and (A4) hold. Thus we get the conclusion by Theorem 3.1. □

In the case that the Assumption (A4) is not satisfied, then instead of (1.4), we can consider

$$\begin{cases} w'(t) = \overline{B_0}w(t) + \int_0^t K(t-s)\overline{B_0}w(s)\,\mathrm{d}s + f(t), & t \geqslant 0, \\ w(0) = w_0, \end{cases} \tag{3.40}$$

whose solution is defined in a way similar to that of (1.4). Now, under the assumption (A3), we know from [5] that $\overline{B_0}$ generates a semigroup $\{S(t)\}_{t \geqslant 0}$ satisfying (2.4)–(2.6), and the solutions of (3.40) are given by

$$w(t) = S(t)w_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s, \quad w_0 \in D(\overline{B_0}). \tag{3.41}$$

That is, we have the same settings as before, thus, the arguments made above for solutions of (1.3) and (1.4) can also be made for solutions of (1.3) and (3.40). Therefore, we have

Theorem 3.3. *Let $T > 0$ be fixed, and* (A1)–(A3), (A5)–(A7) *hold. Let $u(t;\varepsilon)$ and $w(t)$ be the solution of* (1.3) *and* (3.40) *on* $[0, T]$, *respectively. Then*

$$u(t;\varepsilon) \to w(t) \text{ uniformly for } t \in [0, T] \text{ as } \varepsilon \to 0.$$

Remark 3.4. Clearly, if $K(\cdot) \equiv 0$, then $F(\cdot) \equiv 0$, and hence $\hat{f}(t;\varepsilon) = f(t;\varepsilon)$, $\hat{f}(t) = f(t)$. Thus (3.6) and (3.7) give the generalized (i.e., mild) solution of (1.1) and (1.2), respectively. Therefore, when $K(\cdot) \equiv 0$, Theorem 3.3 goes back to Theorems 3.3 and 8.4 in Fattorini [5] for equations without the integral term. Furthermore it is easy to see that if $A = 0$, then (A4) holds due to the definition of $\overline{B_0}$ and (A1). So Theorems 3.1 and 3.3 cover Theorem 2.1 in Liu [10].

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