NONLINEAR IMPULSIVE EVOLUTION EQUATIONS

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Abstract. We study the existence and uniqueness of mild and classical solutions for a nonlinear impulsive evolution equation

$$\begin{aligned}
&u'(t) = Au(t) + f(t, u(t)), \;\; 0 < t < T_0, \;\; t \neq t_i, \\
&u(0) = u_0, \\
&\Delta u(t_i) = I_i(u(t_i)), \;\; i = 1, 2, ..., \;\; 0 < t_1 < t_2 < ... < T_0,
\end{aligned}$$

in a Banach space X, where A is the generator of a strongly continuous semigroup, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, and I_i's are some operators. The impulsive conditions can be used to model more physical phenomena than the traditional initial value problems $u(0) = u_0$. We apply the semigroup theory to first study the existence and uniqueness of the mild solutions, and then show that the mild solutions give rise to classical solutions if f is continuously differentiable.

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1 Introduction.

Recently, the differential equations with impulsive conditions have been studied quite extensively. In which case, the traditional initial value problems

$$u(0) = u_0, \tag{1.1}$$

are replaced by the impulsive conditions

$$u(0) = u_0, \quad \Delta u(t_i) = I_i(u(t_i)), \ \ i = 1, 2, ... \tag{1.2}$$

where $0 < t_1 < t_2 < ...,\ \Delta u(t_i) = u(t_i^+) - u(t_i^-),\ i = 1, 2, ...,$ and I_i's are some operators.

That is, the impulsive conditions are the combinations of the traditional initial value problems and the short-term perturbations whose duration can be negligible in comparison with the duration of the process. They have advantage over the traditional initial value problems because they can be used to model other phenomena that cannot be modeled by the traditional initial value problems, such as the dynamics of populations subjected to abrupt changes (harvesting, diseases, etc.). See Bainov, Kamont and Minchev [1], Chan, Ke and Vatsala [2], Guo and Liu [3], Liu and Willms [4], Rogovchenko [6], Zavalishchin [7] and the references therein for more comments and citations.

For equations in finite dimensional spaces, or equations in general Banach spaces but with bounded (or continuous) operators, the problems have been examined, see, e.g., Guo and Liu [3], Liu and Willms [4], where the existence and uniqueness of solutions (or other type of solutions like extremal solutions), and stability and other properties are studied.

For the evolution equation with an unbounded operator A of the form

$$u'(t) = Au(t) + f(t, u(t)), \ \ t > 0, \ \ t \neq t_i, \tag{1.3}$$

$$u(0) = u_0, \tag{1.4}$$

$$\Delta u(t_i) = I_i(u(t_i)), \ \ i = 1, 2, ..., \tag{1.5}$$

where A is a sectorial operator, Rogovchenko [6] studied the existence and uniqueness of the classical solutions by the successive approximations, with some conditions given on the fractional operators A^α, $\alpha \geq 0$.

The purpose of this paper is to study the existence and uniqueness of mild and classical solutions of the evolution equation (1.3)–(1.5) on $[0, T_0)$ with a general unbounded operator A, (say, e.g., non-sectorial, so that the arguments in Rogovchenko [6] don't apply,) which generates a strongly continuous semigroup $T(\cdot)$. And we will look at the problem using a semigroup approach.

First, we give conditions so as to prove the existence and uniqueness of the mild solutions, given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)). \quad (1.6)$$

Then, we follow the techniques in Pazy [5] to verify that when f is continuously differentiable, the mild solutions give rise to classical solutions.

2 The Mild and Classical Solutions.

Consider the evolution equation

$$u'(t) = Au(t) + f(t,u(t)), \;\; 0 < t < T_0, \;\; t \neq t_i, \quad (2.1)$$

$$u(0) = u_0, \quad (2.2)$$

$$\Delta u(t_i) = I_i(u(t_i)), \;\; i = 1, 2, ..., p, \quad (2.3)$$

in a Banach space $(X, \|\cdot\|)$, where $0 < t_1 < t_2 < ... < t_p < T_0 < \infty$, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$.

Let $PC([0,T_0],X) = \{u : u$ is a map from $[0,T_0]$ into X such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, ..., p\}$. Same as in [3], we see that $PC([0,T_0],X)$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{t\in[0,T_0]} \|u(t)\|. \quad (2.4)$$

We first study the existence and uniqueness of mild solutions using the fixed point argument, under the following assumptions:

(H1). $f : [0,T_0] \times X \to X$ and $I_i : X \to X$, $i = 1, 2, ..., p$, are continuous and there exist constants $L > 0$, $h_i > 0$, $i = 1, 2, ..., p$, such that

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|, \;\; t \in [0,T_0], \; u, v \in X, \quad (2.5)$$

$$\|I_i(u) - I_i(v)\| \leq h_i\|u - v\|, \quad u, v \in X. \quad (2.6)$$

(H2). Let $T(\cdot)$ be the strongly continuous semigroup generated by the unbounded operator A, [5]. Let $B(X)$ be the Banach space of all linear and bounded operators on X. Denote

$$M \equiv \max_{t\in[0,T_0]} \|T(t)\|_{B(X)}, \tag{2.7}$$

then

$$M\Big[LT_0 + \sum_{i=1}^{p} h_i\Big] < 1. \tag{2.8}$$

Note that from the semigroup properties and the Uniform Boundedness Principle, $\|T(t)\|_{B(X)}$ is bounded on $[0, T_0]$. So M in Assumption (H2) is finite.

Under these assumptions, we can prove the following result.

Theorem 2.1. Let Assumptions (H1) – (H2) be satisfied. Then for every $u_0 \in X$, the equation for $t \in [0, T_0]$

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \tag{2.9}$$

has a unique solution.

Proof. Let $u_0 \in X$ be fixed. Define an operator Q on $PC([0, T_0], X)$ by

$$\begin{aligned}(Qv)(t) \quad &= \quad T(t)u_0 + \int_0^t T(t-s)f(s,v(s))ds \\ &\quad + \sum_{0<t_i<t} T(t-t_i)I_i(v(t_i)), \quad 0 \le t \le T_0. \end{aligned} \tag{2.10}$$

Then it is clear that $Q : PC([0, T_0], X) \to PC([0, T_0], X)$. Also we have from Assumption (H1),

$$\begin{aligned} &\|(Qv)(t) - (Qw)(t)\| \\ &\le \int_0^t \|T(t-s)\|_{B(X)}\|f(s,v(s)) - f(s,w(s))\|ds \\ &\quad + \sum_{0<t_i<t} \|T(t-t_i)\|_{B(X)}\|I_i(v(t_i)) - I_i(w(t_i))\| \qquad (2.11) \\ &\le MLT_0\|v-w\|_{PC} + \sum_{0<t_i<t} Mh_i\|v(t_i) - w(t_i)\| \qquad (2.12) \end{aligned}$$

$$\begin{aligned}
&\leq \quad MLT_0\|v-w\|_{PC}+M\|v-w\|_{PC}\sum_{i=1}^{p}h_i &(2.13)\\
&\leq \quad M\Big[LT_0+\sum_{i=1}^{p}h_i\Big]\|v-w\|_{PC}, \quad v,\ w\in PC([0,T_0],X). &(2.14)
\end{aligned}$$

Now from Assumption (H2), we find that Q is a contraction operator on $PC([0,T_0],X)$. This completes the proof. $\square$

Following the semigroup theory, the solutions of (2.9) are called the mild solutions of Eq.(2.1)–(2.3).

Remark 2.1. In cases where I_i's are constants, one has $h_i = 0, i = 1, 2, ..., p$. So we only need $MLT_0 < 1$ in Assumption (H2).

Next, we study the classical solutions. We first give the definition.

Definition 2.1. A classical solution of Eq.(2.1)–(2.3) is a function $u(\cdot) \in PC([0,T_0],X)\cap C^1((0,T_0)\setminus\{t_1,\, t_2,\, ...,\, t_p\},X)$, $u(t)\in D(A)$ (the domain of A) for $t\in(0,T_0)\setminus\{t_1,\, t_2,\, ...,\, t_p\}$, which satisfies Eq.(2.1)–(2.3) on $[0,T_0)$.

Note that the classical solutions for evolution equations without the impulsive conditions are defined in an obvious way. ([5]).

To be able to apply the method in Pazy [5], we also need the following lemmas here.

Lemma 2.1. [5] Consider the evolution equation

$$u'(t) = Au(t)+f(t,u(t)), \quad t_0<t<T_0, \tag{2.15}$$

$$u(t_0) = u_0. \tag{2.16}$$

If $u_0\in D(A)$, and $f(\cdot)\in C^1((t_0,T_0)\times X,X)$, then it has a unique classical solution, which satisfies

$$u(t) = T(t-t_0)u_0+\int_{t_0}^{t}T(t-s)f(s,u(s))ds, \quad t\in[t_0,T_0). \tag{2.17}$$

Lemma 2.2. Let Assumptions (H1) − (H2) be satisfied, and assume that $u_0\in D(A)$ and that $f\in C^1((0,T_0)\times X,X)$. Then for the unique classical solution $u(\cdot) = u(\cdot,u_0)$ on $[0,t_1)$ of Eq.(2.1)–(2.2) without the impulsive conditions (guaranteed by Lemma 2.1), one can define $u(t_1)$ in such a way that $u(\cdot)$ is left continuous at t_1 and $u(t_1) \in D(A)$. (Note: $t_1 < T_0$.)

Proof. Consider the following evolution equation without the impulsive condition on $(0, T_0)$,

$$w'(t) = Aw(t) + f(t, w(t)), \;\; 0 < t < T_0, \tag{2.18}$$

$$w(0) = u_0. \tag{2.19}$$

By Lemma 2.1, it has a classical solution given by

$$w(t) = T(t)u_0 + \int_0^t T(t-s)f(s, w(s))ds, \quad t \in [0, T_0), \tag{2.20}$$

and $w(t) \in D(A)$ for $t \in [0, T_0)$.

Next, applying Lemma 2.1 to $u(\cdot)$, one has, for $t \in [0, t_1) \subset [0, T_0)$,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds, \;\; t \in [0, t_1). \tag{2.21}$$

Now, define

$$u(t_1) = T(t_1)u_0 + \int_0^{t_1} T(t_1 - s)f(s, u(s))ds, \tag{2.22}$$

so that $u(\cdot)$ is left continuous at t_1. Then, apply Lemma 2.1 or the proof of Theorem 2.1 on $[0, t_1]$ to get

$$u(t) = w(t), \;\; t \in [0, t_1]. \tag{2.23}$$

Thus

$$u(t_1) = w(t_1) \in D(A). \qquad \Box \tag{2.24}$$

Before we study the classical solutions of Eq.(2.1)–(2.3), we first prove the following lemma.

Lemma 2.3. Assume that $u_0 \in D(A)$, $q_i \in D(A)$, $i = 1, 2, ..., p$, and that $f \in C^1((0, T_0) \times X, X)$. Then the impulsive equation

$$u'(t) = Au(t) + f(t, u(t)), \;\; 0 < t < T_0, \;\; t \neq t_i, \tag{2.25}$$

$$u(0) = u_0, \tag{2.26}$$

$$\Delta u(t_i) = q_i, \;\; i = 1, 2, ..., p, \tag{2.27}$$

has a unique classical solution $u(\cdot)$ which satisfies, for $t \in [0, T_0)$,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds + \sum_{0<t_i<t} T(t-t_i)q_i. \tag{2.28}$$

Proof. On $J_1 = [0, t_1)$, Lemma 2.1 implies that the equation

$$u'(t) = Au(t) + f(t, u(t)), \ \ 0 < t < t_1, \ \ u(0) = u_0, \tag{2.29}$$

has a unique classical solution $u_1(\cdot)$ which satisfies

$$u_1(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u_1(s))ds, \quad t \in [0, t_1). \tag{2.30}$$

Now, define

$$u_1(t_1) = T(t_1)u_0 + \int_0^{t_1} T(t_1 - s)f(s, u_1(s))ds, \tag{2.31}$$

so that $u_1(\cdot)$ is left continuous at t_1, and $u_1(t_1) \in D(A)$ using Lemma 2.2. Next, on $J_2 = [t_1, t_2)$, consider the equation

$$u'(t) = Au(t) + f(t, u(t)), \ \ t_1 < t < t_2, \ \ u(t_1) = u_1(t_1) + q_1. \tag{2.32}$$

Since $u_1(t_1) + q_1 \in D(A)$, we can use Lemma 2.1 again to get a unique classical solution $u_2(\cdot)$ satisfying for $t \in [t_1, t_2)$

$$u_2(t) = T(t - t_1)\Big[u_1(t_1) + q_1\Big] + \int_{t_1}^t T(t-s)f(s, u_2(s))ds. \tag{2.33}$$

Now, define $u_2(t_2)$ accordingly so that $u_2(\cdot)$ is left-continuous at t_2.

It is easily seen that Lemma 2.2 can be applied to interval $[t_1, T_0)$ to verify that $u_2(t_2) \in D(A)$. It is also easily seen that this procedure can be repeated on $J_k = [t_{k-1}, t_k), \ k = 3, 4, ..., p+1$ (here we need to define $t_{p+1} = T_0$) to get classical solutions

$$u_k(t) = T(t - t_{k-1})\Big[u_{k-1}(t_{k-1}) + q_{k-1}\Big] + \int_{t_{k-1}}^t T(t-s)f(s, u_k(s))ds,$$

for $t \in [t_{k-1}, t_k)$, with $u_i(\cdot)$ left continuous at t_i and $u_i(t_i) \in D(A), \ i = 1, 2, ..., p$.

Now, define

$$u(t) = \begin{cases} u_1(t), & 0 \le x \le t_1, \\ u_k(t), & t_{k-1} < t \le t_k, \quad k = 2, 3, ..., p, \\ u_{p+1}(t), & t_p < t < t_{p+1} = T_0. \end{cases}$$

It is clear that $u(\cdot)$ is the unique classical solution of Eq.(2.25)-(2.27).

Next, we use induction to show that (2.28) is satisfied on $[0, T_0)$. First, (2.28) is satisfied on $[0, t_1]$. If (2.28) is satisfied on $(t_{k-1}, t_k]$, then for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} u(t) &= u_{k+1}(t) = T(t-t_k)\Big[u_k(t_k) + q_k\Big] + \int_{t_k}^{t} T(t-s) f(s, u_{k+1}(s)) ds \\ &= T(t-t_k)\Big[T(t_k)u_0 + \int_0^{t_k} T(t_k - s) f(s, u(s)) ds \\ &\quad + \sum_{0<t_i<t_k} T(t_k - t_i) q_i + q_k\Big] + \int_{t_k}^{t} T(t-s) f(s, u_{k+1}(s)) ds \quad (2.34) \\ &= T(t-t_k)T(t_k)u_0 + \int_0^{t_k} T(t-s) f(s, u(s)) ds + \sum_{0<t_i<t_k} T(t-t_i) q_i \\ &\quad + T(t-t_k) q_k + \int_{t_k}^{t} T(t-s) f(s, u(s)) ds \quad (2.35) \\ &= T(t)u_0 + \int_0^t T(t-s) f(s, u(s)) ds + \sum_{0<t_i<t} T(t-t_i) q_i. \quad (2.36) \end{aligned}$$

Thus (2.28) is also true on $(t_k, t_{k+1}]$. Therefore (2.28) is true on $[0, T_0)$. □

Now, we are in a position to verify that mild solutions give rise to classical solutions if $f \in C^1((0, T_0) \times X, X)$.

Theorem 2.2. Let Assumptions (H1) – (H2) be satisfied, and let $u(\cdot) = u(\cdot, u_0)$ be the unique mild solution of Eq.(2.1)–(2.3) guaranteed by Theorem 2.1. Also assume that $u_0 \in D(A)$, $I_i(u(t_i)) \in D(A)$, $i = 1, 2, ..., p$, and that $f \in C^1((0, T_0) \times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of Eq.(2.1)–(2.3).

Proof. Let $u(\cdot)$ be the mild solution. We can now define $q_i = I_i(u(t_i))$, $i = 1, 2, ..., p$. Then from Lemma 2.3, Eq.(2.25)–(2.27) has a unique classical solution $w(\cdot)$ which satisfies for $t \in [0, T_0)$

$$w(t) = T(t)u_0 + \int_0^t T(t-s)f(s, w(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)). \quad (2.37)$$

Now, $u(\cdot)$ is the mild solution of Eq.(2.1)–(2.3), so that using (2.9) we get for $t \in [0, T_0]$,

$$w(t) - u(t) = \int_0^t T(t-s)\Big[f(s, w(s)) - f(s, u(s))\Big]ds. \quad (2.38)$$

Then the proof of Theorem 2.1 can be applied to show that $u(\cdot) = w(\cdot)$. This implies that $u(\cdot)$ is also a classical solution. □

Remark 2.2. Another way to prove Theorem 2.2 is to directly show that the mild solution is continuously differentiable on $(t_k, t_{k+1}),\ k = 1, 2, ..., p$, using Pazy [5]'s method.

Remark 2.3. From the original proof in Pazy [5], we find that to be able to use Lemma 2.1 to get differentiable solutions, we need, for $j = 1, 2, ..., p$,

$$T(t-t_j)u(t_j^+) = T(t-t_j)\Big[u(t_j) + I_j(u(t_j))\Big] \in D(A),\ \ t > t_j. \quad (2.39)$$

When the operator A generates a general strongly continuous semigroup $T(\cdot)$, we need to prove that $u(t_j) \in D(A)$. So that (2.39) is true with the assumption that $I_j(u(t_j)) \in D(A)$. When the operator A generates an analytical semigroup $T(\cdot)$, then from the semigroup properties,

$$T(t)X \subset D(A),\ \ t > 0. \quad (2.40)$$

So that (2.39) is satisfied automatically. Therefore, we have

Theorem 2.3. Let Assumptions (H1) – (H2) be satisfied, and let $u(\cdot) = u(\cdot, u_0)$ be the unique mild solution of Eq.(2.1)–(2.3) guaranteed by Theorem 2.1. Assume further that the semigroup $T(\cdot)$ is an analytic semigroup, and that $f \in C^1((0, T_0) \times X, X)$. Then for any $u_0 \in X$, $u(\cdot) = u(\cdot, u_0)$ gives rise to a unique classical solution of Eq.(2.1)–(2.3).

Remark 2.4. Theorem 2.3 above gives the same results about the existence and the uniqueness as in Theorem 2.2 of Rogovchenko [6], but with different assumptions and approaches.

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