ASYMPTOTIC PROPERTIES VIA AN INTEGRODIFFERENTIAL INEQUALITY

James H. Liu*

Abstract

Uniform boundedness, ultimate boundedness, uniform stability, and asymptotic stability are studied by analyzing a Liapunov function satisfying

$$v'(t) \leq -\alpha v(t) + \sqrt{v(t)} \int_{\#}^{t} \omega(t,s)\sqrt{v(s)}ds, \ \ t \geq t_0 \geq 0, \ \ (\# = 0 \text{ or } -\infty).$$

The results are then applied to integrodifferential equations

$$x'(t) = A(t)\Big[x(t) + \int_{\#}^{t} F(t,s)x(s)ds\Big], \ \ t \geq t_0 \geq 0, \ \ (\# = 0 \text{ or } -\infty),$$

in real Hilbert space with unbounded linear operators $A(\cdot)$, which occur in viscoelasticity and in heat conduction for materials with memory.

1 INTRODUCTION.

In qualitative studies of differential or integrodifferential equations, Liapunov or Liapunov - Razumikhin methods are very effective in analyzing the asymptotic properties, which include boundedness, ultimate boundedness, stability and asymptotic stability.

It is customary to require that the derivative of a Liapunov function or functional along a solution be negative all the time, or Razumikhin conditions are imposed so that the derivative is negative when the Liapunov function reaches its maximum at t on $[0,t]$ or on $(-\infty,t]$, [3, 4].

---

* Department of Mathematics. James Madison University, Harrisonburg, VA 22807. Liu@math.jmu.edu
AMS Subject Classification : 45J, 45N, 34D.
Key Words : Integrodifferential equations. Boundedness and stability. Liapunov function.

Recently, Hara, Yoneyama and Miyazaki [6] presented some new general results about the asymptotic properties of an integrodifferential equation in $\Re^n$ in which the condition on a Liapunov function is such that

$$v'(t) \leq -\alpha v(t) + \int_0^t \omega(t,s)v(s)ds, \quad t \geq t_0 \geq 0, \tag{1.1}$$

with

$$\limsup_{t\to\infty} \int_0^t \omega(t,s)ds < \alpha \quad \text{and} \quad \lim_{u\to\infty} \int_0^t \omega(u,s)ds = 0 \ \text{ for } t > 0. \tag{1.2}$$

It can be seen that $v'(t) < 0$ if $v(t)$ reaches its maximum on $[0,t]$ for large t. So it is in the right spirit of a Razumikhin condition. However, due to the special forms of inequalities (1.1)-(1.2), the proof of the results are very simple and elementary.

We are interested in the asymptotic properties of integrodifferential equations in real Hilbert spaces, for example, equations of the forms

$$x'(t) = A(t)\Big[x(t) + \int_0^t F(t,s)x(s)ds\Big], \ t \geq t_0 \geq 0, \ \ x(s) = \phi(s), \ 0 \leq s \leq t_0, \tag{1.3}$$

and

$$x'(t) = A(t)\Big[x(t) + \int_{-\infty}^t F(t,s)x(s)ds\Big], \ t \geq t_0 \geq 0, \ \ x(s) = \phi(s), \ s \leq t_0, \tag{1.4}$$

with unbounded linear operators $A(\cdot)$, which can be used to model heat conduction or viscoelasticity for materials with memory. See Grimmer and Liu [3] for the motivation for Eqs.(1.3)–(1.4) and other comments.

Similar to [3] we can construct a Liapunov function $v(\cdot)$ for Eqs.(1.3) and (1.4) in a natural way. Deriving an inequality in the spirit of (1.1), we end up with

$$v'(t) \leq -\alpha v(t) + \sqrt{v(t)} \int_{\#}^t \omega(t,s)\sqrt{v(s)}ds, \ \ t \geq t_0 \geq 0, \ \ (\# = 0 \text{ or } -\infty) \tag{1.5}$$

where ω is determined by F.

Formally, we can convert (1.5) into inequality (1.1), but condition (1.2) is not satisfied. Thus results in [6] cannot be applied to Eq.(1.3) or (1.4) directly. However, (1.5) is "similar" to (1.1), so we will demonstrate here that the techniques used in [6] can be modified to obtain the asymptotic properties under condition (1.5).

We will prove results in uniform stability and asymptotic stability, as well as uniform boundedness and ultimate boundedness, which are not studied in [6]. The results are then applied to Eqs.(1.3) and (1.4). An example in [6] indicates that in general, uniform asymptotic stability is not expected under conditions (1.1) and (1.2). See [5] for a study of uniform asymptotic stability with additional conditions.

We remark that similar results about stability and asymptotic stability for Eqs.(1.3) and (1.4) with $A(t) = A, t \geq 0, F(t,s) = F(t-s)$ are given in Grimmer and Liu [3] by Razumikhin techniques. We will see that the treatment here is much simpler, only elementary differential inequality is used. And results in uniform boundedness and ultimate boundedness can also be obtained in a unified way.

Finally, note that for equation

$$x'(t) = Ax(t) + \int_{\#}^{t} F(t,s)x(s)ds, \quad t \geq t_0 \geq 0, \quad (\# = 0 \text{ or } -\infty) \tag{1.6}$$

in $\Re^n$ with inner product $\langle , \rangle$, we can define $v = \langle x, x \rangle = \|x\|^2$, then

$$\begin{aligned} v'(t) &= 2\langle x(t), x'(t) \rangle \\ &= 2\langle x(t), Ax(t) + \int_{\#}^{t} F(t,s)x(s)ds \rangle \\ &\leq 2\langle x(t), Ax(t) \rangle + 2\sqrt{v(t)} \int_{\#}^{t} \|F(t,s)\| \sqrt{v(s)} ds. \end{aligned} \tag{1.7}$$

If A is a negative definite or a stable matrix, then inequality (1.5) can also occur naturally in this situation. Therefore the study of the asymptotic properties of (1.6) can also be carried out using (1.5).

2 A LEMMA.

In this section we prove a lemma which will be used in the next section to study the asymptotic properties under condition (1.5).

Lemma 2.1. Let $\alpha > 0$ be a constant. Assume that $\omega(t,s) \geq 0$ is continuous for $0 \leq s \leq t$, with

$$\sup_{t \geq 0} \int_{\#}^{t} \omega(t,s)ds < \alpha \quad \text{and} \quad \lim_{u \to \infty} \int_{\#}^{t} \omega(u,s)ds = 0 \ \text{ for } t > 0. \ (\# = 0 \text{ or } -\infty) \tag{2.1}$$

Consider all functions $v(\cdot) = v(\cdot, t_0) : [0, \infty) \to [0, \infty)$, or $(-\infty, \infty) \to [0, \infty)$, such that

$$v'(t) \le -\alpha v(t) + \sqrt{v(t)} \int_{\#}^{t} \omega(t,s)\sqrt{v(s)}ds, \quad t \ge t_0 \ge 0. \tag{2.2}$$

(a). If $v(s) = v(s, t_0) \le M$ for $0 \le s \le t_0$ when $\# = 0$, or for $s \le t_0$ when $\# = -\infty$, then $v(t) \le M,\ t \ge t_0$.

(b). For any $B > 0$, $B_0 > 0$ and $t_0 \ge 0$, there is a constant $T = T(B, B_0, t_0) > 0$ such that if $v(\cdot) = v(\cdot, t_0) \ge 0$ is a function satisfying (2.2) with $v(s) \le B_0$, here $0 \le s \le t_0$ when $\# = 0$ or $s \le t_0$ when $\# = -\infty$, then $v(t) < B,\ t \ge T + t_0$.

Proof. (a): We only prove for $\# = 0$ since the proof for $\# = -\infty$ is the same. Assume that $v(s) \le M$, $0 \le s \le t_0$. If $\{v(t) \le M,\ t \ge t_0\}$ is not true, then there is $M_1 > M$ and $t_1 > t_0$ such that $v(t_1) = M_1$ and $v(t) \le M_1,\ t \in [0, t_1]$. Now from (2.2) we have for $t_0 \le t \le t_1$,

$$v(t) \le v(t_0)e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-r)}\Big(\sqrt{v(r)} \int_0^r \omega(r,s)\sqrt{v(s)}ds\Big)dr. \tag{2.3}$$

So that

$$\begin{aligned} M_1 &= v(t_1) \le Me^{-\alpha(t_1-t_0)} + M_1\alpha \int_{t_0}^{t_1} e^{-\alpha(t_1-r)}dr & (2.4)\\ &< M_1\Big[e^{-\alpha(t_1-t_0)} + \alpha \int_{t_0}^{t_1} e^{-\alpha(t_1-r)}dr\Big] = M_1, & (2.5) \end{aligned}$$

which is a contradiction.

(b): Again we only prove for $\# = 0$ since the proof for $\# = -\infty$ is the same. If the result is not true, then there exist $B > 0$, $B_0 > 0$ and $t_0 \ge 0$, and sequence $\{v_k(\cdot) = v_k(\cdot, t_0)\}$ satisfying (2.2) and $t_k \to \infty$, as $k \to \infty$, such that $v_k(s) \le B_0,\ 0 \le s \le t_0$ and $v_k(t_k) \ge B$.

Accordingly, we can define nonempty sets

$$\begin{aligned} P = \Big\{&\text{sequence } \{u_k(\cdot) = u_k(\cdot, t_0)\} \text{ on } [0,\infty) \mid \{u_k(\cdot)\} \text{ satifies (2.2), and there are}\\ &s_k \to \infty \text{ as } k \to \infty, \text{ such that } u_k(s) \le B_0,\ 0 \le s \le t_0 \text{ and } u_k(s_k) \ge B\Big\}, \end{aligned} \tag{2.6}$$

and

$$P^* = \Big\{\limsup_{k\to\infty} u_k(s_k) \mid \{u_k(\cdot)\} \in P\Big\}. \tag{2.7}$$

Now, from result (a), $u_k(s) \le B_0,\ 0 \le s \le t_0$ implies $u_k(s_k) \le B_0$ for $s_k \in [0, \infty)$. Then $P^* \subseteq [B, B_0]$. Thus

$$\infty > L \equiv \max\{p \mid p \in P^*\} \ge B > 0.$$

From (2.1), there is $0 < \gamma < \alpha$ with

$$\int_0^t \omega(t,s)ds < \gamma,\ t \ge 0. \tag{2.8}$$

Then there is θ with $\frac{\gamma}{\alpha} < \theta < 1$. As $\frac{(\alpha\theta+\gamma)L}{2\theta\alpha} < L$, there is $\{u_k(\cdot)\} \in P$ such that

$$\limsup_{k\to\infty} u_k(s_k) > \frac{(\alpha\theta+\gamma)L}{2\theta\alpha}. \tag{2.9}$$

From the definition of L, it is easily seen that for this $\{u_k(\cdot)\} \in P$, there is $H > 0$ such that

$$u_k(t) \le \frac{L}{\theta}, \quad k \ge H,\ t \ge s_H. \tag{2.10}$$

Now from (2.1) we can find $t^* > \max\{H, s_H, t_0\}$ such that

$$\int_0^{s_H} \omega(u,s)ds < \frac{(\alpha\theta-\gamma)L}{2\theta B_0},\ \ u \ge t^*. \tag{2.11}$$

Thus from (2.2), (2.8), (2.10) and (2.11), one has for $k \ge H,\ t \ge t^*$, (note that from result (a), $u_k(s) \le B_0,\ 0 \le s \le t_0$ implies $u_k(t) \le B_0, t \ge 0$)

$$\begin{aligned}
u_k(t) \ &\le\ u_k(t^*)e^{-\alpha(t-t^*)} + \int_{t^*}^t e^{-\alpha(t-r)}\Big(\sqrt{u_k(r)}\int_0^{s_H}\omega(r,s)\sqrt{u_k(s)}ds\Big)dr && (2.12)\\
&\quad + \int_{t^*}^t e^{-\alpha(t-r)}\Big(\sqrt{u_k(r)}\int_{s_H}^r \omega(r,s)\sqrt{u_k(s)}ds\Big)dr && (2.13)\\
&\le\ B_0e^{-\alpha(t-t^*)} + B_0\int_{t^*}^t e^{-\alpha(t-r)}\Big(\int_0^{s_H}\omega(r,s)ds\Big)dr && (2.14)\\
&\quad + \frac{L}{\theta}\int_{t^*}^t e^{-\alpha(t-r)}\Big(\int_0^r \omega(r,s)ds\Big)dr && (2.15)\\
&\le\ B_0e^{-\alpha(t-t^*)} + \frac{(\alpha\theta-\gamma)L}{2\theta\alpha} + \frac{\gamma L}{\theta\alpha} && (2.16)\\
&\le\ B_0e^{-\alpha(t-t^*)} + \frac{(\alpha\theta+\gamma)L}{2\theta\alpha}. && (2.17)
\end{aligned}$$

Let k be large so that $s_k > t^*$. Hence from (2.9) and (2.17),

$$\frac{(\alpha\theta+\gamma)L}{2\theta\alpha} < \limsup_{k\to\infty} u_k(s_k) \le \frac{(\alpha\theta+\gamma)L}{2\theta\alpha},$$

which is a contradiction. $\square$

Remark 2.2. If $\omega(t,s) = \omega(t-s)$, then condition (2.1) is equivalent to $\int_0^\infty \omega(s)ds < \alpha$.

3 THE ASYMPTOTIC PROPERTIES.

In this section, we will study the asymptotic properties under conditions (2.1)–(2.2). The results include uniform boundedness, ultimate boundedness, uniform stability and asymptotic stability. They can be applied to any differential or integrodifferential equations for which a Liapunov function satisfying (2.1)–(2.2) can be constructed.

First, for convenient references, we give the following standard definitions for the case $\# = 0$. The definitions for $\# = -\infty$ are stated accordingly. Note that in the following, we use "system" to denote any differential or integrodifferential equation in a space with norm $\|\cdot\|$.

Definition 3.1. For any $t_0 \geq 0$ and any continuous function ϕ on $[0, t_0]$, a solution $u(\cdot, t_0, \phi)$ of a "system" is a function on $[0, \infty)$ satisfying the "system" for $t \geq t_0$ and $u(s) = \phi(s)$ for $s \in [0, t_0]$.

Definition 3.2. Solutions $u(\cdot) = u(\cdot, t_0, \phi)$ of a "system" are uniformly bounded if for each $B_1 > 0$ there is $B_2 = B_2(B_1) > 0$ such that $\{\|\phi(s)\| \leq B_1,\ 0 \leq s \leq t_0\}$ imply $\|u(t)\| < B_2,\ t \geq t_0$.

Definition 3.3. Solutions $u(\cdot) = u(\cdot, t_0, \phi)$ of a "system" are ultimate bounded if there is a bound $B > 0$ such that for each $B_3 > 0$ and $t_0 \geq 0$ there is $T = T(B, B_3, t_0) > 0$ such that $\{\|\phi(s)\| \leq B_3,\ 0 \leq s \leq t_0\}$ imply $\|u(t)\| < B,\ t \geq T + t_0$.

Definition 3.4. Assume $u \equiv 0$ is a solution of a "system". Then solution $u \equiv 0$ is stable if given $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\{\|\phi(s)\| < \delta$ on $[0, t_0]$ and $u(t, t_0, \phi)$ being a solution of the system$\}$ imply $\|u(t, t_0, \phi)\| < \varepsilon$ for $t \geq t_0$. It is uniformly stable if it is stable and the δ is independent of t_0.

Definition 3.5. Assume $u \equiv 0$ is a solution of a "system". Then solution $u \equiv 0$ is asymptotically stable if it is stable and for any $t_0 \geq 0$ there exists a constant $r = r(t_0) > 0$ such that $\{\|\phi(s)\| < r$ on $[0, t_0]$ and $u(t, t_0, \phi)$ being a solution of the system$\}$ imply $u(t, t_0, \phi) \to 0$, as $t \to \infty$.

Applying Lemma 2.1, we now have the following result.

Theorem 3.6. Assume that there exist functions $W_i, i = 1, 2$, with $W_i : [0, \infty) \to [0, \infty)$ and W_1 strictly increasing. (Known as wedges.) Further, assume that there exists a function V (known as a Liapunov function) such that for solutions $u(\cdot)$ of a "system",

(c1). $W_1(\|u(t)\|) \leq V(u(t)) \leq W_2(\|u(t)\|)$,

(c2). $v(t) \equiv V(u(t))$ satisfies (2.1)–(2.2).

Then solutions of the "system" are uniformly bounded and ultimate bounded, and zero solution $u \equiv 0$ is uniformly stable and asymptotic stable.

Proof. Again, we only prove for $\# = 0$ since the proof for $\# = -\infty$ is the same. From the definition of W_i and condition (c1), we only need to prove the corresponding statements for $v(t) \equiv V(u(t))$.

Uniform boundedness: For $B_1 > 0$, choose $B_2 = B_1$. Then from Lemma 2.1(a), $v(s) \leq B_1$, $0 \leq s \leq t_0$ implies $v(t) \leq B_1 = B_2, t \geq t_0$.

Ultimate boundedness: Choose $B = 1$. Then from Lemma 2.1(b), for any B_3 (treat it as B_0 in Lemma 2.1(b)) > 0 and $t_0 \geq 0$, there is $T = T(B, B_3, t_0) > 0$ such that $v(s) \leq B_3$, $0 \leq s \leq t_0$ implies $v(t) < B$, $t \geq T + t_0$.

Uniformly stability: Given $\epsilon > 0$, choose $\delta(\epsilon) = \epsilon$. Then from Lemma 2.1(a), $\|v(s)\| \leq \delta(\epsilon) = \epsilon$, $0 \leq s \leq t_0$ implies $\|v(t)\| \leq \epsilon,\ t \geq t_0$.

Asymptotic stability: Stability is proven already. Next, for any $t_0 \geq 0$, choose $r = r(t_0) = 1$. Then from Lemma 2.1(b), for any ϵ (treat it as B in Lemma 2.1(b)) > 0, $B_0 \equiv r(= 1)$, there is $T = T(\epsilon, 1, t_0) = T(\epsilon, t_0) > 0$ such that $v(s) \leq B_0 = r$, $0 \leq s \leq t_0$ implies $v(t) \leq \epsilon$, $t \geq T + t_0$. (Thus $v(t) \to 0$, $t \to \infty$.) $\square$

4 INTEGRODIFFERENTIAL EQUATIONS.

In this section, we will apply the above results to

$$x'(t) = A(t)\Big[x(t) + \int_0^t F(t,s)x(s)ds\Big],\ \ t \geq t_0 \geq 0,\ \ x(s) = \phi(s),\ 0 \leq s \leq t_0, \tag{4.1}$$

and

$$x'(t) = A(t)\Big[x(t) + \int_{-\infty}^t F(t,s)x(s)ds\Big],\ \ t \geq t_0 \geq 0,\ \ x(s) = \phi(s),\ s \leq t_0, \tag{4.2}$$

with unbounded operators $A(\cdot)$ in real Hilbert space X with inner product $\langle , \rangle$. Since we only study asymptotic properties here, we will assume the existence and uniqueness of solutions, which can be found in, e.g., [1, 2, 7]. We define a Liapunov function V for $z = (x, w) \in X \times X$ by

$$V(z) = \langle x, x\rangle - 2\langle x, w\rangle + \frac{3}{2}\langle w, w\rangle. \tag{4.3}$$

Also we define $\|z\|^2 \equiv \|x\|^2 + \|w\|^2$ and let $z(t) = (x(t), w(t))$ with $x(\cdot)$ a solution of Eq.(4.1) or Eq.(4.2) and

$$w(t) = x(t) + \int_{\#}^{t} F(t, s)x(s)ds, \quad t \geq 0 \quad (\# = 0 \text{ or } -\infty).$$

Then it is clear that in order to prove the asymptotic properties of solutions $x(\cdot)$ of Eq.(4.1) or Eq.(4.2), we only need to prove the corresponding statements on $z(\cdot)$.

Theorem 4.1. Suppose that solutions $x(\cdot)$ of Eqs.(4.1) and (4.2) exist and are unique on $[0, \infty)$ (when initial data satisfy certain conditions), and suppose that for some constants $\lambda > 0$ and $\beta > 0$,

$$\langle A(t)x, x\rangle \leq -\lambda\langle x, x\rangle, \;\; x \in D(A(t)), \;\; t \geq 0, \tag{4.4}$$

and

$$\langle F(t, t)x, x\rangle \geq \beta\langle x, x\rangle, \;\; x \in X, \;\; t \geq 0,$$

where D means domain. Then

(a). $\|z\|^2/5 \leq V(z) \leq 3\|z\|^2$,

(b). $v(t) \equiv V(z(t))$ satisfies (2.1)–(2.2) with $\# = 0$ for Eq.(4.1), or with $\# = -\infty$ for Eq.(4.2), where

$$\alpha = \min_{t\geq 0} \frac{1}{3}\Big\{\lambda - \frac{3}{2}\|F(t, t)\|, \; 2\beta - \frac{3}{2}\|F(t, t)\|\Big\},$$

and

$$\omega(t, s) = (6 + 3\sqrt{6})\|\frac{\partial}{\partial t}F(t, s)\|.$$

So that if

$$\sup_{t\geq 0} \int_0^t \omega(t, s)ds < \alpha \quad \text{and} \quad \lim_{u\to\infty} \int_0^t \omega(u, s)ds = 0 \;\; \text{for } t \geq 0,$$

then solutions of Eq.(4.1) are uniformly bounded and ultimate bounded, and zero solution of Eq.(4.1) is uniformly stable and asymptotic stable.

Similarly, if

$$\sup_{t\geq 0}\int_{-\infty}^{t}\omega(t,s)ds<\alpha \quad \text{and} \quad \lim_{u\to\infty}\int_{-\infty}^{t}\omega(u,s)ds=0 \ \text{ for } t\geq 0,$$

then solutions of Eq.(4.2) are uniformly bounded and ultimate bounded, and zero solution of Eq.(4.2) is uniformly stable and asymptotic stable.

Proof. First, we have

$$\begin{aligned} V(z) &\geq \|x\|^2-2\|x\|\|w\|+\frac{3}{2}\|w\|^2 \\ &= (\|x\|-\|w\|)^2+\frac{1}{2}\|w\|^2 \\ &= \frac{1}{6}(3\|w\|-2\|x\|)^2+\frac{1}{3}\|x\|^2. \end{aligned} \tag{4.5}$$

Thus we obtain $\|z\|^2/5\leq V(z)\leq 3\|z\|^2$. Next, differentiating $v(t)\equiv V(z(t))$ with respect to t yields (note that $*$ denotes convolution)

$$\begin{aligned} v'(t) &= \frac{d}{dt}V(z(t))=2\langle x'(t),x(t)\rangle-2\langle x'(t),w(t)\rangle-2\langle w'(t),x(t)\rangle+3\langle w'(t),w(t)\rangle \\ &= \langle A(t)w(t),w(t)\rangle-2\langle F(t,t)x(t),x(t)\rangle-2\langle x(t),\frac{\partial}{\partial t}F*x(t)\rangle \\ &\quad +3\langle F(t,t)x(t),w(t)\rangle+3\langle\frac{\partial}{\partial t}F*x(t),w(t)\rangle \\ &\leq -\lambda\|w(t)\|^2-2\beta\|x(t)\|^2+3\|F(t,t)\|\|x(t)\|\|w(t)\| \\ &\quad +\Big(2\|x(t)\|+3\|w(t)\|\Big)\|\frac{\partial}{\partial t}F*x(t)\| \\ &\leq -\lambda\|w(t)\|^2-2\beta\|x(t)\|^2+\frac{3}{2}\|F(t,t)\|\Big(\|x(t)\|^2+\|w(t)\|^2\Big) \\ &\quad +\Big(2\sqrt{3}+3\sqrt{2}\Big)\sqrt{v(t)}\|\frac{\partial}{\partial t}F*x(t)\| \quad \text{(from (4.5))} \\ &\leq \Big(-\lambda+\frac{3}{2}\|F(t,t)\|\Big)\|w(t)\|^2+\Big(-2\beta+\frac{3}{2}\|F(t,t)\|\Big)\|x(t)\|^2 \\ &\quad +\Big(2\sqrt{3}+3\sqrt{2}\Big)\sqrt{v(t)}\|\frac{\partial}{\partial t}F*x(t)\| \\ &\leq -3\alpha\Big(\|x(t)\|^2+\|w(t)\|^2\Big)+\Big(2\sqrt{3}+3\sqrt{2}\Big)\sqrt{v(t)}\|\frac{\partial}{\partial t}F*x(t)\| \\ &\leq -\alpha v(t)+\Big(2\sqrt{3}+3\sqrt{2}\Big)\sqrt{v(t)}\Big(\sqrt{3}\|\frac{\partial}{\partial t}F\|*\sqrt{v}(t)\Big) \quad \text{(from } (a)\text{)} \end{aligned}$$

$$\leq \quad -\alpha v(t) + \sqrt{v(t)}\Big(6+3\sqrt{6}\Big)\Big(\|\frac{\partial}{\partial t}F\| * \sqrt{v}(t)\Big), \tag{4.6}$$

where

$$\|\frac{\partial}{\partial t}F\| * \sqrt{v}(t) = \int_0^t \|\frac{\partial}{\partial t}F(t,s)\|\sqrt{v(s)}ds \quad \text{and} \quad \int_{-\infty}^t \|\frac{\partial}{\partial t}F(t,s)\|\sqrt{v(s)}ds$$

for Eqs.(4.1) and (4.2) respectively. $\square$

Remark 4.2. It is known that $A(t) \equiv \frac{\partial^2}{\partial x^2}, t \geq 0$ with domain $H_0^1(0,1) \cap H^2(0,1)$ satisfy (4.4) on $X = L^2(0,1)$ with $\lambda = 1$. Thus applications can be carried out. We omit them here for simplicity.

References

[1] G. Da Prato and E. Sinestrari, *Non autonomous evolution operators of hyperbolic type*, Semigroup Forum, **45**(1992), 302-321.

[2] R. Grimmer, *Resolvent operators for integral equations in a Banach space*, Transactions of Amer. Math. Soc., **273**(1982), 333-349.

[3] R. Grimmer and J. Liu, *Liapunov - Razumikhin methods for integrodifferential equations in Hilbert space*, Delay and Differential Equations, A. Fink, R. Miller and W. Kliemann (eds.), World Scientific, London, 1992, 9-24.

[4] R. Grimmer and G. Seifert, *Stability Properties of Volterra Integrodifferential Equations*, J. Diff. Eq., **19**(1975), 142 - 166.

[5] T. Hara, T. Yoneyama and T. Itoh, *Asymptotic stability criteria for nonlinear Volterra integro-differential equations*, Funkcial. Ekvac., **33** (1990), 39-57.

[6] T. Hara, T. Yoneyama and R. Miyazaki, *Volterra integro-differential inequality and asymptotic criteria*, Diff. Integ. Eq., **1**(1992), 201-212.

[7] J. Liu, *Integrodifferential equations with non autonomous operators*, submitted.

[8] G. Seifert, *Liapunov - Razumikhin Conditions for Asymptotic Stability in Functional Differential Equations of Volterra Type*, J. Diff. Eq., **16**(1974), 289 - 297.