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SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CAUCHY PROBLEM

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1. INTRODUCTION

Consider the following semilinear Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \quad 0 \le t \le T, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

in a Banach space X with A the generator of a strongly continuous semigroup $T(\cdot)$. Equations (1.1)–(1.2) have been studied by many authors. For example, Pazy [1] studied equations (1.1)–(1.2) by first showing the existence and uniqueness of mild solutions, that is, solutions of

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))\,\mathrm{d}s, \quad 0 \le t \le T, \tag{1.3}$$

when $f(t, u)$ is Lipschitz in u. A fixed point theorem was used in the proof. Then the mild solutions were proven to be classical solutions if $f \in C^1([0, T] \times X, X)$.

Motivated by physical problems, Byszewski [2], Byszewski and Lakshmikantham [3], Jackson [4] and references therein generalized the "classical" Cauchy problem (1.1)–(1.2) to the following nonlocal Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \quad 0 \le t \le T, \tag{1.4}$$

$$u(0) + g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) = u_0, \tag{1.5}$$

where $0 < t_1 < t_2 < \cdots t_p \le T$. For example, $g(t_i, u(t_i))$ may be given by

$$g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) = \sum_{i=1}^{p} c_i u(t_i), \tag{1.6}$$

where c_i $(i = 1, \ldots, p)$ are given constants. In this case, (1.5) allows the measurements at $t = 0, t_1, \ldots, t_p$, rather than just at $t = 0$. So more information is available. Thus equations (1.4)–(1.5) can be applied in physics with better effect than equations (1.1)–(1.2). See [2–4] and references therein for more comments.

In Byszewski [2], techniques employed in Pazy [1] were generalized in the study of (1.4)–(1.5). That is, existence and uniqueness of solutions (called mild solutions) of

$$u(t) = T(t)\Big[u_0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))\Big] + \int_0^t T(t-s) f(s, u(s))\, ds, \quad 0 \le t \le T, \tag{1.7}$$

was first proved, using a fixed point argument, when $f(t, u)$ is Lipschitz in u. Then the mild solutions were shown to be classical solutions if $f \in C^1([0, T] \times X, X)$.

Now, look at the following classical heat equation for material with memory, see, e.g. [5],

$$\begin{cases} q(t, x) = -E u_x(t, x) - \displaystyle\int_0^t b(t-s) u_x(s, x)\, ds, \\ u_t(t, x) = -\partial q(t, x)/\partial x + f(t, x), \\ u(0, x) = u_0(x). \end{cases} \tag{1.8}$$

The first equation gives the heat flux and the second is the balance equation. Equation (1.8) can be written as (assuming $E = 1$)

$$u_t(t, x) = \frac{\partial^2}{\partial x^2}\Big[u(t, x) + \int_0^t b(t-s) u(s, x)\, ds\Big] + f(t, x), \quad u(0, x) = u_0(x). \tag{1.9}$$

It is clear that if nonlocal condition (1.5) is introduced to (1.9), then it will also have better effect than the classical condition $u(0, x) = u_0(x)$, since the same comments as above apply here. Therefore, we would like to extend the above results for (1.4)–(1.5) to the following semilinear integrodifferential equation with nonlocal Cauchy problem

$$u'(t) = A\Big[u(t) + \int_0^t F(t-s) u(s)\, ds\Big] + f(t, u(t)), \quad 0 \le t \le T, \tag{1.10}$$

$$u(0) + g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) = u_0, \tag{1.11}$$

in a Banach space X with A the generator of a strongly continuous semigroup and $F(t)$ a bounded operator for $t \in [0, T]$. (For example, in (1.9), $A = \partial^2/\partial x^2$ on $H^2(0, 1) \cap H_0^1(0, 1)$ generates a strongly continuous semigroup on $L^2(0, 1)$.) Since formulas (1.3) and (1.7) played very important roles in the studies of (1.1)–(1.2) and (1.4)–(1.5), we are looking for a similar formula

$$u(t) = R(t)\Big[u_0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))\Big] + \int_0^t R(t-s) f(s, u(s))\, ds, \quad 0 \le t \le T, \tag{1.12}$$

where the semigroup $T(\cdot)$ in (1.3) and (1.7) is replaced by the resolvent operator $R(\cdot)$, the counterpart of $T(\cdot)$ for integrodifferential equations.

The existence, uniqueness, representation of solutions via variation of constants formula, and other properties of resolvent operators have been studied, e.g. in [6–8], so we are able to use the techniques developed in Pazy [1] and Byszewski [2] to study (1.10)–(1.11). Namely, we will first show the existence and uniqueness of solutions of (1.12) using a fixed point result (they are called mild solutions). Then mild solutions are shown to be classical solutions if $f \in C^1([0, T] \times X, X)$.

In Section 2, we will provide some basic results about resolvent operator $R(\cdot)$, including the representation of solutions via the variation of constants formula. In Section 3, we will study the nonlocal Cauchy problem (1.10)–(1.11). Finally, in Section 4, we study the special case when $\|R(t)\|_{B(X)} \leq Me^{-\alpha t}$, $0 \leq t \leq T$ for some constant $\alpha > 0$ and when the nonlocal condition (1.11) is given by (1.6). We will see that in this case, conditions in assumption (H5) in Section 3 can be improved.

The study in this paper also has the following feature: assume that f is T_0–periodic in t for some $T_0 > 0$ fixed and $u(\cdot)$ is a solution of (1.10). Then $w(t) \equiv u(t + T_0)$, $t \geq 0$, does not satisfy (1.10), if $F \neq 0$. So it was remarked in, e.g. [9, p.88] that (1.10) has no T_0–periodic solutions except for a "few" examples that periodic solutions do exist. (See [9] for more comments.) In nonlocal conditions (1.11) and (1.6), if $u_0 = 0$, $p = 1$, $t_1 = T_0$, $c_1 = -1$, then one has $u(0) = u(T_0)$. So our results indicate that although (1.10) has no T_0–periodic solutions, it has a solution u on $[0, T]$ such that $u(0) = u(T_0)$, $T_0 \leq T$. We may call such solutions "local periodic solutions". Note that they may not be T_0–periodic.

2. RESOLVENT OPERATORS

In this section, we collect some basic results about resolvent operators from [7,8]. We will make the following assumptions

(H1) A generates a strongly continuous semigroup in Banach space X,

(H2) $F(t) \in B(X)$, $0 \leq t \leq T$. $F(t) : Y \to Y$ and for $x(\cdot)$ continuous in Y, $AF(\cdot)x(\cdot) \in L^1([0, T], X)$. For $x \in X$, $F'(t)x$ is continuous in $t \in [0, T]$, where $B(X)$ is the space of all linear and bounded operators on X, and Y is the Banach space formed from $D(A)$, the domain of A, endowed with the graph norm. Observe that in many applications, $F(\cdot)$ is a scalar or an appropriate matrix, so (H2) is satisfied there. Now we give the following definitions.

Definition 2.1 [8]. $R(\cdot)$ is a resolvent operator of (1.10) with $f \equiv 0$ if $R(t) \in B(X)$ for $0 \leq t \leq T$ and satisfies

(1) $R(0) = I$ (the identity operator on X),

(2) for all $u \in X$, $R(t)u$ is continuous for $0 \leq t \leq T$,

(3) $R(t) \in B(Y)$, $0 \leq t \leq T$. For $y \in Y$, $R(\cdot)y \in C^1([0, T], X) \cap C([0, T], Y)$ and

$$\frac{d}{dt}R(t)y = A\Big[R(t)y + \int_0^t F(t-s)R(s)y\, ds\Big]$$

$$= R(t)Ay + \int_0^t R(t-s)AF(s)y\, ds, \quad 0 \leq t \leq T. \tag{2.1}$$

Definition 2.2. $u(\cdot, u_0) \in C([0, T], X)$ is a mild solution of (1.10)–(1.11) if it satisfies

$$u(t) = R(t)\Big[u_0 - g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p))\Big]$$

$$+\int_0^t R(t-s)f(s,u(s))\,ds, \quad 0 \le t \le T. \tag{2.2}$$

Definition 2.3. A classical solution of (1.10)–(1.11) is a function $u(\cdot) \in C([0,T],Y) \cap C^1([0,T],X)$ which satisfies (1.10)–(1.11) on $[0,T]$. We denote it by $u(\cdot,u_0)$.

The existence and uniqueness of resolvent operators is guaranteed by the following result.

THEOREM 2.4. Let Assumptions (H1) and (H2) be satisfied. Then (1.10) with $f \equiv 0$ has a unique resolvent operator.

Proof. Consider (1.10)–(1.11) with $f = 0$, $g = 0$. Then theorem 2.2 in [7] shows that assumptions (H1) and (H2) imply the existence of a resolvent operator $R(\cdot)$. Next, note that when $f = 0$, $g = 0$ and $u_0 \in D(A)$, $R(\cdot)u_0$ is the unique classical solution of (1.10)–(1.11). So if $R_1(\cdot)$ is another resolvent operator of (1.10) with $f \equiv 0$, then for $t_0 \in [0,T]$ fixed, $R_1(t_0) = R(t_0)$ on $D(A)$. However, $D(A)$ is dense in X, so $R_1(t_0) = R(t_0)$ on X. This proves the uniqueness of $R(\cdot)$. ■

We also need the following result of [8, theorem 4.5] concerning the classical solutions of the linear Cauchy problem when $f(t,u) = f(t)$ and $g = 0$ in (1.10)–(1.11).

THEOREM 2.5 [8]. Let assumptions (H1) and (H2) be satisfied and assume that $f(t,u) = f(t)$, $g = 0$, $u_0 \in D(A)$, and $f(\cdot) \in C^1([0,T],X)$. Then (1.10)–(1.11) has a unique classical solution.

We close this section by stating the variation of constants formula given in, e.g. [6,8].

THEOREM 2.6. Let $f \in C([0,T] \times X, X)$ and let $R(\cdot)$ be a resolvent operator for (1.10) with $f \equiv 0$. If u is a classical solution of (1.10)–(1.11), then it satisfies

$$u(t) = R(t)u(0) + \int_0^t R(t-s)f(s,u(s))\,ds, \quad 0 \le t \le T. \tag{2.3}$$

3. NONLOCAL CAUCHY PROBLEM

In this section, we will use the techniques developed in Pazy [1] and Byszewski [2] to study (1.10)–(1.11). We first study the existence and uniqueness of mild solutions using fixed point argument, under the following assumptions

(H3) $f : [0,T] \times X \to X$ is continuous in $t \in [0,T]$ and there exists a constant $L > 0$ such that

$$\|f(t,u) - f(t,v)\|_X \le L\|u - v\|_X, \quad t \in [0,T], \ u, v \in X, \tag{3.1}$$

(H4) $g : [0,T]^p \times X^p \to X$ and there exists a constant $K > 0$ such that

$$\begin{aligned}&\|g(t_1,\ldots,t_p,u(t_1),\ldots,u(t_p)) - g(t_1,\ldots,t_p,v(t_1),\ldots,v(t_p))\|_X \\ &\le K\|u - v\|_{C([0,T],X)},\end{aligned} \tag{3.2}$$

(H5) denote

$$M \equiv \max_{t\in[0,T]} \|R(t)\|_{B(X)}, \tag{3.3}$$

then

$$M(K+TL) < 1. \tag{3.4}$$

Remark 3.1. If $F = 0$ in (1.10), then resolvent operator of (1.10) with $f \equiv 0$ and the semigroup generated by A are the same. Next, note that from definition 2.1 (2) and the uniform boundedness principle, $\|R(t)\|_{B(X)}$ is bounded on $[0, T]$. So assumption (H5) makes sense. Also note that if the inequality in assumption (H3) holds only for u, v in a small ball in X, as posed in Byszewski [2], then similar conditions as those in [2] can be stated here. For simplicity, we assume that (3.1) is true on X.

Under these assumptions, we can prove the existence and uniqueness of mild solutions.

THEOREM 3.2. Let assumptions (H1) – (H5) be satisfied. Then for every $u_0 \in X$, (1.10)–(1.11) has a unique mild solution.

Proof. Let $u_0 \in X$ be fixed. Define an operator $Q : C([0,T],X) \to C([0,T],X)$ by

$$\begin{aligned}(Qv)(t) = R(t)\Big[u_0 - g(t_1,\dots,t_p,v(t_1),\dots,v(t_p))\Big]\\ + \int_0^t R(t-s)f(s,v(s))\,\mathrm{d}s, \quad 0 \le t \le T.\end{aligned} \tag{3.5}$$

Then

$$\begin{aligned}&\|(Qv)(t) - (Qw)(t)\|_X\\ &\le \|R(t)\|_{B(X)}\|g(t_1,\dots,t_p,v(t_1),\dots,v(t_p)) - g(t_1,\dots,t_p,w(t_1),\dots,w(t_p))\|_X\\ &\quad + \int_0^t \|R(t-s)\|_{B(X)}\|f(s,v(s)) - f(s,w(s))\|_X\,\mathrm{d}s\\ &\le MK\|v-w\|_{C([0,T],X)} + M\int_0^t L\|v(s)-w(s)\|_X\,\mathrm{d}s\\ &\le M(K+TL)\|v-w\|_{C([0,T],X)}, \quad v,\ w \in C([0,T],X),\ t\in[0,T].\end{aligned} \tag{3.6}$$

Now, from assumption (H5), we find that Q is a contraction operator on $C([0,T],X)$. This completes the proof. ■

Next, we prove that mild solutions are classical solutions if $f \in C^1([0,T]\times X, X)$.

THEOREM 3.3. Let assumptions (H1) – (H5) be satisfied and let $u(\cdot)$ be the unique mild solution of (1.10)–(1.11) guaranteed by theorem 3.2. Assume further that $u_0 \in D(A)$, $g : [0,T]^p \times X^p \to D(A)$ and that $f \in C^1([0,T]\times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of

(1.10)–(1.11).

Proof. Since all conditions of theorem 3.2 are satisfied, we may denote by $u(\cdot) = u(\cdot, u_0)$ the unique mild solution of (1.10)–(1.11) such that $u(0) = u_0 - g(t_1, \dots, t_p, u(t_1), \dots, u(t_p))$. We will show that $u(\cdot) \in C^1([0, T], X)$. To this end, we set

$$B(s) \equiv \frac{\partial}{\partial u} f(s, u), \quad s \in [0, T], \tag{3.7}$$

and

$$k(t) \equiv R(t) f(0, u(0)) + A\Big[R(t)u(0) + \int_0^t F(t-s)R(s)u(0)\, \mathrm{d}s\Big] + \int_0^t R(t-s) \frac{\partial}{\partial s} f(s, u(s))\, \mathrm{d}s, \quad t \in [0, T]. \tag{3.8}$$

Note that now $u(0) \in Y$, so from definition 2.1 and our assumptions, $k(\cdot) \in C([0, T], X)$. Thus the method used in Pazy [1, pp. 184–187] or in the proof of theorem 3.2 can be used here to show that

$$w(t) = k(t) + \int_0^t R(t-s)B(s)w(s)\, \mathrm{d}s, \quad t \in [0, T], \tag{3.9}$$

has a unique solution $w(\cdot) \in C([0, T], X)$. Moreover, from our assumptions we have

$$f(s, u(s+h)) - f(s, u(s)) = B(s)\Big[u(s+h) - u(s)\Big] + \omega_1(s, h), \tag{3.10}$$

and

$$f(s+h, u(s+h)) - f(s, u(s+h)) = \frac{\partial}{\partial s} f(s, u(s+h))h + \omega_2(s, h), \tag{3.11}$$

where

$$h^{-1}\|\omega_i(s, h)\| \to 0, \quad h \to 0, \tag{3.12}$$

uniformly on $s \in [0, T]$ for $i = 1, 2$. Define

$$w_h(t) \equiv \frac{u(t+h) - u(t)}{h} - w(t), \quad t \in [0, T]. \tag{3.13}$$

Then from (3.7), (3.8), (3.9), (3.13) and the fact that $u(\cdot)$ is a mild solution, we obtain

$$w_h(t) = \Big(h^{-1}\Big[R(t+h)u(0) - R(t)u(0)\Big] - A\Big[R(t)u(0) + \int_0^t F(t-s)R(s)u(0)\, \mathrm{d}s\Big]\Big)$$

$$+h^{-1}\int_0^t R(t-s)\Big(\omega_1(s,h)+\omega_2(s,h)\Big)\,\mathrm{d}s$$

$$+\int_0^t R(t-s)\Big(\frac{\partial}{\partial s}f(s,u(s+h))-\frac{\partial}{\partial s}f(s,u(s))\Big)\,\mathrm{d}s$$

$$+\Big(h^{-1}\int_0^h R(t+h-s)f(s,u(s))\,\mathrm{d}s-R(t)f(0,u(0))\Big)$$

$$+\int_0^t R(t-s)B(s)w_h(s)\,\mathrm{d}s. \tag{3.14}$$

From the definition of resolvent operator and our assumptions, it is clear that the norm of each one of the four first terms on the right-hand side of (3.14) tends to zero as $h \to 0$. Therefore, we have

$$\|w_h(t)\|_X \le \epsilon(h) + M_*\int_0^t \|w_h(s)\|_X\,\mathrm{d}s, \tag{3.15}$$

where

$$M_* = \max_{t\in[0,T]} \|R(t-s)B(s)\|_{B(X)}, \tag{3.16}$$

and

$$\epsilon(h) \to 0, \quad h \to 0. \tag{3.17}$$

From (3.15) it follows by Gronwall's inequality that

$$\|w_h(t)\|_X \le \epsilon(h)\mathrm{e}^{TM_*}, \quad t \in [0,T] \tag{3.18}$$

and, therefore,

$$\|w_h(t)\|_X \to 0, \quad h \to 0, \quad t \in [0,T]. \tag{3.19}$$

This implies that $u(t)$ is differentiable on $[0,T]$ and that its derivative is $w(t)$. Since $w(\cdot) \in C([0,T],X)$, $u(\cdot) \in C^1([0,T],X)$.

Finally, to show that $u(\cdot)$ is the classical solution of (1.10)–(1.11) we note that since $u(\cdot) \in C^1([0,T],X)$ and $f \in C^1([0,T]\times X, X)$, $t \to f(t,u(t))$ is in $C^1([0,T],X)$. Therefore, theorems 2.5 and 2.6 imply that the linear Cauchy problem

$$v'(t) = A\Big[v(t) + \int_0^t F(t-s)v(s)\,\mathrm{d}s\Big] + f(t,u(t)), \quad 0 \le t \le T, \tag{3.20}$$

$$v(0) = u_0 - g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)), \tag{3.21}$$

has a unique classical solution $v(\cdot)$ given by

$$v(t) = R(t)\Big[u_0 - g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p))\Big] + \int_0^t R(t-s)f(s, u(s))\, ds, \quad 0 \le t \le T. \tag{3.22}$$

However, the right-hand side of (3.22) is just $u(t)$ since $u(\cdot)$ is the mild solution. So we have $v(t) = u(t)$, $t \in [0, T]$ and, hence, $u(\cdot)$ is the classical solution of (1.10)–(1.11). This proves the result. ■

4. WHEN $\|R(T)\|_{B(X)} \le ME^{-\alpha T}$, $0 \le T \le T$, $\alpha > 0$

In this section, we study the special case when $\|R(t)\|_{B(X)} \le Me^{-\alpha t}$, $0 \le t \le T$ for some constant $\alpha > 0$ and when the nonlocal condition (1.11) is given by (1.6). We will see that in this case, conditions in assumption (H5) in Section 3 can be improved. Since we can now first prove the existence and uniqueness of mild solution $u(\cdot, v)$ of Cauchy problem

$$\begin{aligned} u'(t) &= A\Big[u(t) + \int_0^t F(t-s)u(s)\, ds\Big] + f(t, u(t)), \quad 0 \le t \le T, \\ u(0) &= v, \end{aligned} \tag{4.1}$$

for any $v \in X$, and then we are able to define an operator along the trajectory of $u(\cdot, v)$ and show that the operator is a contraction, and finally argue that the fixed point of the operator gives rise to a mild solution of (1.10)–(1.11). We now list the following assumptions

(H6) For some constant $\alpha > 0$, the resolvent operator of (1.10) with $f \equiv 0$ satisfies

$$\|R(t)\|_{B(X)} \le Me^{-\alpha t}, \ 0 \le t \le T. \tag{4.2}$$

(H7) Nonlocal condition (1.11) is given by (1.6) and

$$\beta \equiv \alpha - ML > 0, \ \ M\sum_{i=1}^{p} |c_i| e^{-\beta t_i} < 1. \quad (L \text{ from } (3.1),\ \alpha,\ M \text{ from } (4.2).) \tag{4.3}$$

Remark 4.1. Note that condition (4.3) is better than (3.4) in some situations.

We need the following inequality from [10] in our proof.

LEMMA 4.2 [10]. Let $u(t)$ and $b(t)$ be nonnegative continuous functions for $t \ge \alpha$, and let

$$u(t) \le ae^{-\gamma(t-\alpha)} + \int_\alpha^t e^{-\gamma(t-s)} b(s)u(s)\, ds, \quad t \ge \alpha,$$

where $a \ge 0$ and γ are constants. Then

$$u(t) \le a\exp\Big(-\gamma(t-\alpha) + \int_\alpha^t b(s)\, ds\Big), \quad t \ge \alpha.$$

THEOREM 4.3. Let assumptions (H1) – (H3), (H6) and (H7) be satisfied. Then for every $u_0 \in X$, (1.10)–(1.11) has a unique mild solution.

Proof. Let $u_0 \in X$ be fixed. Then for any $v \in X$, define an operator $Q : C([0,T], X) \to C([0,T], X)$ by

$$(Qu)(t) = R(t)v + \int_0^t R(t-s)f(s, u(s))\,\mathrm{d}s, \quad 0 \le t \le T. \tag{4.4}$$

Then

$$\begin{aligned}
&\|(Qu)(t) - (Qw)(t)\|_X \\
&\le \int_0^t \|R(t-s)\|_{B(X)} \|f(s, u(s)) - f(s, w(s))\|_X \,\mathrm{d}s \\
&\le M \int_0^t L\|u(s) - w(s)\|_X \,\mathrm{d}s \\
&\le MLt\|u - w\|_{C([0,T],X)}, \quad u,\ w \in C([0,T], X),\ t \in [0,T].
\end{aligned} \tag{4.5}$$

Using (4.4), (4.5) and induction on n it follows that

$$\|(Q^n u)(t) - (Q^n w)(t)\|_X \le \frac{(MLt)^n}{n!}\|u - w\|_{C([0,T],X)}, \tag{4.6}$$

hence

$$\|Q^n u - Q^n w\|_{B(X)} \le \frac{(MLT)^n}{n!}\|u - w\|_{C([0,T],X)}. \tag{4.7}$$

For n large enough $(MLT)^n/n! < 1$ and by the extension of the contraction principle, Q has a unique fixed point $u(\cdot, v)$ (which is the mild solution of (4.1)) such that $u(0) = v$. Next, define another operator $Q_1 : X \to X$ by

$$Q_1 v = u_0 - \sum_{i=1}^{p} c_i u(t_i), \tag{4.8}$$

where $u(\cdot) = u(\cdot, v)$ is the unique fixed point of (4.4). Then if we denote for $i = 1, 2$, $u_i(\cdot) = u_i(\cdot, v_i)$, which is the unique fixed point of (4.4) with $u_i(0) = v_i$, we obtain

$$\|Q_1 v_1 - Q_1 v_2\|_X \le \sum_{i=1}^{p} |c_i| \|u_1(t_i) - u_2(t_i)\|_X. \tag{4.9}$$

Now, for $w(\cdot) \equiv u_1(\cdot) - u_2(\cdot)$, we have from (4.4),

$$\|w(t)\|_X \le \|R(t)\|_{B(X)}\|v_1 - v_2\|_X + \int_0^t \|R(t-s)\|_{B(X)}\|f(s, u_1(s)) - f(s, u_2(s))\|_X \,\mathrm{d}s$$

$$\leq M\|v_1 - v_2\|_X e^{-\alpha t} + \int_0^t MLe^{-\alpha(t-s)}\|u_1(s) - u_2(s)\|_X \, ds$$

$$= M\|v_1 - v_2\|_X e^{-\alpha t} + \int_0^t MLe^{-\alpha(t-s)}\|w(s)\|_X \, ds, \quad t \in [0, T].$$

Thus from lemma 4.2,

$$\|w(t)\|_X \leq M\|v_1 - v_2\|_X e^{-(\alpha - ML)t} = M\|v_1 - v_2\|_X e^{-\beta t}, \quad t \in [0, T]. \tag{4.10}$$

Therefore, from (4.10), (4.9) becomes

$$\|Q_1 v_1 - Q_1 v_2\|_X \leq \left(M \sum_{i=1}^{p} |c_i| e^{-\beta t_i}\right) \|v_1 - v_2\|_X. \tag{4.11}$$

From assumption (H7), Q_1 is a contraction operator on X. Thus Q_1 has a unique fixed point $v_0 \in X$. Therefore, for the unique fixed point $u(\cdot, v_0)$ of (4.4) corresponding to v_0, we obtain

$$u(0, v_0) = v_0 = u_0 - \sum_{i=1}^{p} c_i u(t_i, v_0). \tag{4.12}$$

This implies that

$$u(t, v_0) = R(t)\left[u_0 - \sum_{i=1}^{p} c_i u(t_i, v_0)\right] + \int_0^t R(t-s) f(s, u(s, v_0)) \, ds, \quad 0 \leq t \leq T$$

and, hence, $u(\cdot, v_0)$ is a mild solution of (1.10)–(1.11). Finally, we note that mild solutions of (1.10)–(1.11) are unique. Since, if $u(\cdot)$ is a mild solution of (1.10)–(1.11) with (1.11) given by (1.6), then

$$u(0) = u_0 - \sum_{i=1}^{p} c_i u(t_i),$$

and $u(\cdot)$ is also the unique mild solution of (4.1) with $v = u(0)$. However, Q_1 is a contraction operator, so (4.6) implies that $u(0)$ is uniquely determined by Q_1. Next, Q is also a contraction operator, so fixed point of (4.4) is uniquely determined by $v = u(0)$. Thus it is clear that mild solutions of (1.10)–(1.11) are unique. This completes the proof. ■

Similar to theorem 3.3, we have the following result which says that mild solutions are classical solutions if $f \in C^1([0, T] \times X, X)$.

THEOREM 4.4. Let assumptions (H1)–(H3), (H6) and (H7) be satisfied and let $u(\cdot)$ be the unique mild solution of (1.10)–(1.11) guaranteed by theorem 4.3. Assume further that $u_0 \in D(A)$, $\sum_{i=1}^{p} c_i u(t_i) \in D(A)$ and that $f \in C^1([0, T] \times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of (1.10)–(1.11).

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