Periodic Solutions of Non-Densely Defined Delay Evolution Equations

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Abstract

We study finite delay evolution equation

$$\begin{cases} x'(t) = Ax(t) + F(t, x_t), & t \geq 0, \\ x_0 = \varphi \in C\left([-r, 0], E\right), \end{cases}$$

where linear operator A is non-densely defined and satisfies the Hille-Yosida condition. First we obtain some properties of "integral solutions" in this case, and prove the compactness of an operator determined by integral solutions. This allows us to apply Horn's fixed point theorem to prove the existence of periodic integral solutions when integral solutions are bounded and ultimately bounded. This extends the study of periodic solutions for densely defined operators to non-densely defined operators. An example is given.

1 INTRODUCTION.

In this note, we consider finite delay evolution equation

$$\begin{cases} x'(t) = Ax(t) + F(t, x_t), & t \geq 0, \\ x_0 = \varphi \in C\left([-r, 0], E\right), \end{cases} \tag{1.1}$$

where A is a non-densely defined linear operator in a Banach space E; $C = C\left([-r, 0], E\right)$ is the space of continuous functions from $[-r, 0]$ ($r > 0$ is a constant) to E endowed with the super-norm; and for every $t \geq 0$, the function $x_t \in C$ is defined by

$$x_t(\cdot)(\theta) = x_t(\theta) = x(t + \theta), \quad \theta \in [-r, 0],$$

and $F(t, \varphi)$ is a continuous function from $\Re \times C$ to E, and is $\omega-$periodic in t.

It's well known that if A is the infinitesimal generator of a C_0-semigroup of bounded linear operators, or equivalently

$(i).\ \overline{D(A)} = E, \quad (D \text{ means domain}),$
$(ii).\ \text{there exist } M \geq 0 \text{ and } \tau \in \Re \text{ such that }]\tau, \infty[\subset \rho(A), \text{ and}$
$\sup\left\{(\lambda - \tau)^n \left|(\lambda - A)^{-n}\right| : \ \lambda > \tau,\ n \in N\right\} \leq M,$

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where $\rho(A)$ is the resolvent set of A, then the classical semigroup theory ensures the well-posedness of Eq.(1.1). ([12]).

In this work we investigate the case when the operator A satisfies only the assumption (ii) above, that is, when A is non-densely defined; and we are concerned with the existence of periodic solutions for Eq.(1.1). The related studies and examples concerning non-densely defined operators can be found in the references such as [1, 3, 4, 7]. For example, in [3], the authors proved that Eq.(1.1) is wellposed in the set

$$C_0 = \left\{ \varphi \in C\left([-r, 0], E \right) : \varphi(0) \in \overline{D(A)} \right\}.$$

The problem of finding periodic solutions is an important subject in the qualitative study of ordinary and functional differential equations. The famous Massera's theorem [9] on two-dimensional periodic ordinary differential equations explains the relationship between the bounded solutions and periodic solutions. Using Browder's fixed point theorem, Yoshizawa proved in [13] that if the solutions of an $n-$dimensional periodic ordinary differential equation are either uniformly bounded or uniformly ultimately bounded, then the system has a periodic solution. Hale and Lopes [6] used Horn's fixed point theorem to obtain the same result for $n-$dimensional periodic ordinary and functional differential equations with finite delay. In [11], Show studied the Massera's theorem for some functional differential equations in finite dimensional spaces and proved that the existence of a bounded solution implies the existence of a periodic solution.

In many of those studies, the most important feature is to show that the operator

$$P\phi = x_\omega(\cdot, \phi), \quad (\omega \text{ units along } x) \tag{1.2}$$

is compact (continuous and takes a bounded set into a precompact set), where ω is the period of the system and x is the unique solution determined by ϕ. Then, some fixed point theorem can be used to derive periodic solutions.

For equations in general Banach spaces (infinite dimensional), showing the compactness of operator P is a very hard task, because the difficulty involving the abstract version of Arzela-Ascoli's theorem. As can be seen in [2, 8, 14], if this difficulty can be overcome, then other steps for equations in finite dimensional spaces can be carried over to show that operator P is compact, and hence to derive periodic solutions. For example, in [8], the author treated the evolution equation

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t), x_t), & t \geq 0, \\ x_0 = \varphi \in C\left([-r, 0], E \right), \end{cases}$$

where $A(t)$ is $\omega-$periodic in t and generates an evolution system $(U(t, s))_{t \geq s \geq 0}$ in a Banach space E, and $f(t, ., .)$ is $\omega-$periodic in t; and proved the compactness of P and hence the existence of periodic mild solutions using the boundedness and ultimate boundedness of mild solutions when $\omega > r$.

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When operator A in Eq.(1.1) satisfies only (ii) above, that is, when A is non-densely defined, the appropriate solutions to work with will be integral solutions; see Definition 2.1. To our knowledge, we have not seen results about periodic integral solutions, thus we would like to provide one here. Similar to the cases for densely defined operators mentioned above, the most important step here is to show that $P\phi = x_\omega(\cdot, \phi)$ is compact, where x is the unique integral solution determined by ϕ. After this, we are able to apply Horn's fixed point theorem and prove, under the boundedness and ultimate boundedness assumptions, the existence of $\omega-$periodic integral solutions for Eq.(1.1) when $\omega > r$. This way, we can extend the study of periodic solutions for densely defined operators to non-densely defined operators. Finally, we give an example of a partial functional differential equation in continuous functions space with the super-norm.

2   THE COMPACTNESS OF OPERATOR P.

In this section we make the following assumptions.

(H_1). The operator A is a Hille-Yosida operator. That means: there exist $M > 0$ and $\tau \in \Re$ such that $]\tau, \infty[\subset \rho(A)$, and $\sup \{(\lambda - \tau)^n |(\lambda - A)^{-n}| : \lambda > \tau, \ n \in N\} \leq M$.

(H_2). The function F is continuous in t and is locally Lipschitz in the second variable in the sense that for each $h > 0$ there exists a constant $k_1(h) > 0$ such that

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq k_1(h) |\varphi_1 - \varphi_2|, \text{ for } t \in [0, h] \text{ and } |\varphi_1|, |\varphi_2| \leq h.$$

We now list the following definitions.

Definition 2.1. Let $b > 0$. A continuous function $x : [-r, b] \to E$ is called an integral solution of Eq.(1.1) if
  $i)$. $\int_0^t x(s)ds \in D(A)$, for $t \in [0, b]$,
  $ii)$. $x(t) = \varphi(0) + A \int_0^t x(s)ds + \int_0^t F(s, x_s)ds$, for $t \in [0, b]$,
  $iii)$. $x_0 = \varphi$.

Definition 2.2. A continuous function $x : [-r, b] \to E$ is called a strict solution of Eq.(1.1) if
  $i)$. $x \in C^1([0, b], E) \cap C([0, b], D(A))$,
  $ii)$. x satisfies Eq.(1.1) for $t \in [0, b]$,
  $iii)$. $x_0 = \varphi$.

Remark 2.1. From the closedness of operator A, we see that if an integral solution x of Eq.(1.1) is continuously differentiable, then x is a strict solution of Eq.(1.1).

We can use methods similar to those in [3] for autonomous equations to obtain the following existence and uniqueness results concerning integral solutions and strict solutions for non-autonomous equations. Details are omitted here.

Theorem 2.1. Assume that (H_1) and (H_2) hold. Then for any $\varphi \in C_0 = \{\varphi \in C([-r, 0], E) : \varphi(0) \in \overline{D(A)}\}$, there exists a unique integral solution $x(., \varphi)$ of Eq.(1.1)

with its maximal interval of existence $[-r, t_\varphi[$, $t_\varphi > 0$, and

$$\text{either } t_\varphi = \infty, \text{ or } \limsup_{t \to t_\varphi} |x(t, \varphi)| = \infty.$$

Moreover $x(., \varphi)$ is a continuous function of φ in the sense that, for $\varphi \in C_0$ and $t \in [0, t_\varphi[$, there exist positive constants L and ε such that for any $\psi \in C_0$ with $|\varphi - \psi| < \varepsilon$, we have

$$t \in [0, t_\psi[\quad \text{and} \quad |x(s, \varphi) - x(s, \psi)| \leq L \, |\varphi - \psi|, \quad s \in [-r, t].$$

Theorem 2.2. Assume that (H_1) and (H_2) hold. Furthermore assume that F is continuously differentiable, and for each $h > 0$, there exist positive constants $k_2(h)$ and $k_3(h)$ such that

$$\begin{aligned}
|D_1 F(t, \varphi_1) - D_1 F(t, \varphi_2)| &\leq k_2(h) |\varphi_1 - \varphi_2|, \\
|D_2 F(t, \varphi_1) - D_2 F(t, \varphi_2)| &\leq k_3(h) |\varphi_1 - \varphi_2|,
\end{aligned}$$

for $t \in [0, h]$ and $|\varphi_1|, |\varphi_2| \leq h$, where D_1 and D_2 denote respectively the derivative of $F(t, \varphi)$ with respect to t and φ. Let $\varphi \in C_0$ such that

$$\varphi' \in C_0, \ \varphi(0) \in D(A), \text{ and } \varphi'(0) = A\varphi(0) + F(0, \varphi).$$

Then the integral solution $x(\varphi)$ of Eq.(1.1) is a strict solution in $[0, t_\varphi[$.

Remark 2.2. If we assume that function $F(t, \varphi)$ is continuous and Lipschitz with respect to the second argument, then the integral solution of Eq.(1.1) exists for all $t \geq 0$. Moreover the solutions $x(t, \varphi)$ are locally bounded in t and φ. In the following, we're concerned with periodic integral solutions, so we will assume that all integral solutions exist on $[0, \infty)$.

Now, we introduce the part A_0 of A in $\overline{D(A)}$, defined by

$$A_0 = A \quad \text{on} \quad D(A_0) = \left\{ x \in D(A) : Ax \in \overline{D(A)} \right\}.$$

Then it's well known ([1, 7]) that the part A_0 of A generates a strongly continuous semigroup $T_0(\cdot)$ on $\overline{D(A)}$; and for $\varphi \in C_0$, the integral solution of Eq.(1.1) is given by

$$x(t) = \begin{cases} T_0(t)\varphi(0) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s) B_\lambda F(s, x_s) ds, & t \geq 0, \\ \varphi(t), \ t \in [-r, 0], \end{cases} \tag{2.1}$$

where $B_\lambda = \lambda(\lambda - A)^{-1}$; and $x(t) \in \overline{D(A)}$, $t \geq 0$.

In the sequel, we need to prove the compactness of operator P, so we assume

(H_3). The semigroup $(T_0(t))_{t \geq 0}$ is compact on $\overline{D(A)}$. That means for each $t > 0$, operator $T_0(t)$ is compact on $\overline{D(A)}$.

Now, we are ready to prove our main result in this paper, that is, the compactness of operator P.

Theorem 2.3. Assume that (H_1), (H_2), (H_3) hold and that all integral solutions of Eq.(1.1) exist for $t \geq 0$ and are locally bounded (that is, for any $t > 0$, the integral solutions of Eq.(1.1)

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are bounded on $[0, t]$ by a constant if their initial functions are bounded by a constant). Let $\omega > r$ be fixed. Then the operator $P\varphi = x_\omega(., \varphi)$ on C_0 is compact (continuous and takes a bounded set into a precompact set).

Proof: The continuity of P follows from Theorem 2.1. Let B be a bounded set in C_0. Then, as integral solutions are locally bounded, it follows that the set

$$W = \{x_\omega(., \varphi) : \varphi \in B\} \tag{2.2}$$

is bounded in C_0. Next, we will show that W is precompact in C_0 by using Arzela-Ascoli's theorem. For that end we first prove that for every fixed $\theta \in [-r, 0]$,

$$\{x(\omega + \theta, \varphi) : \varphi \in B\}$$

is precompact in $\overline{D(A)}$. From (2.1), we have

$$x(\omega + \theta, \varphi) = T_0(\omega + \theta)\varphi(0) + \lim_{\lambda \to \infty} \int_0^{\omega+\theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds.$$

Since $\omega > r$ is fixed, we can choose $\varepsilon > 0$ such that $\omega + \theta - \varepsilon > 0$. Then

$$\int_0^{\omega+\theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds = \int_0^{\omega+\theta-\varepsilon} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds$$
$$+ \int_{\omega+\theta-\varepsilon}^{\omega+\theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds,$$

and

$$\lim_{\lambda \to \infty} \int_0^{\omega+\theta-\varepsilon} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds =$$
$$T_0(\varepsilon) \lim_{\lambda \to \infty} \int_0^{\omega+\theta-\varepsilon} T_0(\omega + \theta - \varepsilon - s)B_\lambda F(s, x_s(., \varphi))ds.$$

Therefore, by Assumption (H_3) and the local boundedness assumption on integral solutions, we see that

$$\left\{ \lim_{\lambda \to \infty} \int_0^{\omega+\theta-\varepsilon} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds : \varphi \in B \right\}$$
$$= T_0(\varepsilon) \left\{ \lim_{\lambda \to \infty} \int_0^{\omega+\theta-\varepsilon} T_0(\omega + \theta - \varepsilon - s)B_\lambda F(s, x_s(., \varphi))ds : \varphi \in B \right\}$$

is precompact in $\overline{D(A)}$ because $T_0(\varepsilon)$ is applied to a bounded set. Next, for some positive constant a, one has

$$\left| \lim_{\lambda \to \infty} \int_0^{\omega+\theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds - \lim_{\lambda \to \infty} \int_0^{\omega+\theta-\varepsilon} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds \right|$$
$$\leq \left| \lim_{\lambda \to \infty} \int_{\omega+\theta-\varepsilon}^{\omega+\theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds \right| \leq a\varepsilon.$$

This implies that the set

$$\left\{ \lim_{\lambda \to \infty} \int_0^{\omega+\theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds : \varphi \in B \right\} \tag{2.3}$$

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is totally bounded, and therefore the set $\{x(\omega + \theta, \varphi) : \varphi \in B\}$ is precompact in $\overline{D(A)}$.

It remains to prove the equicontinuity of functions in set W. Let $\theta, \theta_0 \in [-r, 0]$ with $\theta > \theta_0$. Then $\omega + \theta > \omega + \theta_0 > 0$, and

$$
\begin{aligned}
x(\omega + \theta, \varphi) - x(\omega + \theta_0, \varphi) \;=\; & [T_0(\omega + \theta) - T_0(\omega + \theta_0)]\varphi(0) \\
& + \lim_{\lambda \to \infty} \int_0^{\omega + \theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds \\
& - \lim_{\lambda \to \infty} \int_0^{\omega + \theta_0} T_0(\omega + \theta_0 - s)B_\lambda F(s, x_s(., \varphi))ds.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\int_0^{\omega + \theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds \;=\; & \int_0^{\omega + \theta_0} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds \\
& + \int_{\omega + \theta_0}^{\omega + \theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds,
\end{aligned}
$$

so we deduce that

$$
\begin{aligned}
|x(\omega + \theta, \varphi) - x(\omega + \theta_0, \varphi)| \;\leq\; & |T_0(\omega + \theta) - T_0(\omega + \theta_0)|\,|\varphi(0)| \\
& + |\lim_{\lambda \to \infty} \int_0^{\omega + \theta_0} [T_0(\omega + \theta - s) - T_0(\omega + \theta_0 - s)]\,B_\lambda F(s, x_s(., \varphi))ds| \\
& + |\lim_{\lambda \to \infty} \int_{\omega + \theta_0}^{\omega + \theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds|.
\end{aligned}
$$

From assumption (H_3) and a result in [10], semigroup $(T_0(t))_{t \geq 0}$ is uniformly continuous for $t > 0$, which implies that

$$
\lim_{\theta \to \theta_0} |T_0(\omega + \theta) - T_0(\omega + \theta_0)| = 0,
$$

since $\omega + \theta > \omega + \theta_0 > 0$. Furthermore, there exists a positive constant b such that

$$
|\lim_{\lambda \to \infty} \int_{\omega + \theta_0}^{\omega + \theta} T_0(\omega + \theta - s)B_\lambda F(s, x_s(., \varphi))ds| \leq b\,(\theta - \theta_0).
$$

We also have

$$
\begin{aligned}
& \lim_{\lambda \to \infty} \int_0^{\omega + \theta_0} [T_0(\omega + \theta - s) - T_0(\omega + \theta_0 - s)]\,B_\lambda F(s, x_s(., \varphi))ds \\
& = [T_0(\theta - \theta_0) - I]\lim_{\lambda \to \infty} \int_0^{\omega + \theta_0} T_0(\omega + \theta_0 - s)B_\lambda F(s, x_s(., \varphi))ds.
\end{aligned}
$$

As in (2.3), we obtain that there is a compact set K in $\overline{D(A)}$ such that

$$
\lim_{\lambda \to \infty} \int_0^{\omega + \theta_0} T_0(\omega + \theta_0 - s)B_\lambda F(s, x_s(., \varphi))ds \in K, \quad \text{for all } \varphi \in B.
$$

Thus, from Banach-Stenhauss's theorem, we have

$$
\lim_{\theta \to \theta_0} \sup_{x \in K} |(T_0(\theta - \theta_0) - I)x| = 0.
$$

This implies that

$$
\lim_{\theta \to \theta_0^+} |x(\omega + \theta, \varphi) - x(\omega + \theta_0, \varphi)| = 0, \quad \text{uniformly for } \varphi \in B.
$$

The proof for $\theta < \theta_0$ is similar. This completes the proof. $\qquad \square$

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3 BOUNDEDNESS AND PERIODICITY.

After showing the compactness of operator P, we can follow [2, 8, 14] and derive periodic integral solutions for Eq.(1.1). In the sequel we make the following assumption.

(H_4) The function $F(t, \varphi)$ is $\omega-$periodic in t with $\omega > r$.

We also need to bound certain terms involving integral solutions, so we define the following.

Definition 3.1. We say that integral solutions of Eq.(1.1) are bounded if for each $B_1 > 0$, there is a $B_2 > 0$ such that $|\varphi| \leq B_1$ implies $|x(t, \varphi)| \leq B_2$ for $t \geq 0$.

Definition 3.2. We say that integral solutions of Eq.(1.1) are ultimate bounded if there is a bound $B > 0$, such that for each $B_3 > 0$, there is a $k > 0$ such that $|\varphi| \leq B_3$ and $t \geq k$ imply $|x(t, \varphi)| \leq B$.

We also list the following result as a reference.

Lemma 3.1. [6] (**Horn's fixed point Theorem**). Let $E_0 \subset E_1 \subset E_2$ be convex subsets of Banach space Z, with E_0 and E_2 compact subsets and E_1 open relative to E_2. Let $P : E_2 \to Z$ be a continuous map such that for some integer m one has

$$P^j(E_1) \subset E_2, \quad 1 \leq j \leq m - 1,$$
$$P^j(E_1) \subset E_0, \quad m \leq j \leq 2m - 1,$$

then P has a fixed point in E_2.

With these preparations, we can prove

Theorem 3.1. Assume that (H_1), (H_2), (H_3) and (H_4) hold. Furthermore suppose that integral solutions of Eq.(1.1) are bounded and ultimate bounded. Then Eq.(1.1) has an $\omega-$periodic integral solution.

Proof: From Theorem 2.3, we know that operator $P\phi = x_\omega(\cdot, \phi)$ on C_0 is compact. Let B be the bound in the definition of ultimate boundedness, then by boundedness, there is a $B_1 > 0$ such that $|\varphi| \leq B$ implies $|x(t, \varphi)| \leq B_1$ for $t \geq 0$. Furthermore, there is a $B_2 > B_1$ such that $|\varphi| \leq B_1$ implies $|x(t, \varphi)| \leq B_2$ for $t \geq 0$. Now, using ultimate boundedness, there is a positive integer m such that $|\varphi| \leq B_1$ implies $|x(t, \varphi)| \leq B$ for $t \geq (m-2)\omega$.

Let $x(t)$ be an integral solution of Eq.(1.1) and define $y(t) = x(t + \omega)$. Then, for $t \geq 0$,

$$
\begin{aligned}
y(t) &= x(t + \omega) = T_0(t + \omega)\phi(0) + \lim_{\lambda \to \infty} \int_0^{t+\omega} T_0(t + \omega - h)B_\lambda F(h, x_h)dh \\
&= T_0(t)T_0(\omega)\phi(0) + \lim_{\lambda \to \infty} \int_0^{\omega} T_0(t + \omega - h)B_\lambda F(h, x_h)dh \\
&\quad + \lim_{\lambda \to \infty} \int_\omega^{t+\omega} T_0(t + \omega - h)B_\lambda F(h, x_h)dh \\
&= T_0(t)\Big[T_0(\omega)\phi(0) + \lim_{\lambda \to \infty} \int_0^{\omega} T_0(\omega - h)B_\lambda F(h, x_h)dh\Big]
\end{aligned}
$$

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$$+ \lim_{\lambda \to \infty} \int_0^t T_0(t-s) B_\lambda F(s, x_{\omega+s}) ds$$

$$= T_0(t) y(0) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s) B_\lambda F(s, y_s) ds. \qquad (3.1)$$

This implies that $y(t) = x(t+\omega)$ is also an integral solution of Eq.(1.1). Then the uniqueness implies that $P^m \varphi = x_{m\omega}(., \varphi)$ for $\varphi \in C_0$. Thus, we have the following

$$|P^{j-1} \varphi| = |x_{(j-1)\omega}(., \varphi)| < B_2, \quad j = 1, 2, ..., m-1 \text{ and } |\varphi| \le B_1,$$
$$|P^{j-1} \varphi| = |x_{(j-1)\omega}(., \varphi)| < B, \quad j \ge m \text{ and } |\varphi| \le B_1.$$

Let

$$H = \{\varphi \in C_0 : |\varphi| < B_2\}, \quad E_2 = \overline{cov.(P(H))},$$
$$K = \{\varphi \in C_0 : |\varphi| < B_1\}, \quad E_1 = K \cap E_2,$$
$$G = \{\varphi \in C_0 : |\varphi| < B\}, \quad E_0 = \overline{cov.(P(G))},$$

where $cov.(F)$ is the convex hull of the set F. Then we see that E_0, E_1 and E_2 are convex subsets of C_0 with E_0 and E_2 compact subsets and E_1 open relative to E_2, and

$$P^j(E_1) \subset P^j(K) = PP^{j-1}(K) \subset P(H) \subset E_2, \quad 1 \le j \le m-1,$$
$$P^j(E_1) \subset P^j(K) = PP^{j-1}(K) \subset P(G) \subset E_0, \quad m \le j \le 2m-1.$$

Consequently, from Horn's fixed point theorem we know that operator P has a fixed point ϕ. Then, for the integral solution $x(\cdot) = x(\cdot, \phi)$ with $x_0(\cdot, \phi) = \phi$, we see from (3.1) that $y(t) = x(t+\omega, \phi)$ is also an integral solution with $y_0 = x_\omega(\phi) = P\phi = \phi$. Thus the uniqueness implies $x(t, \phi) = y(t) = x(t+\omega, \phi)$; therefore $x(t, \phi)$ is an ω-periodic solution. This completes the proof. $\qquad \square$

Next, we give a criteria as in [8] to guarantee the boundedness and ultimate boundedness of integral solutions of Eq.(1.1). The proof can be found in [8].

Theorem 3.2. Assume that there exist functions $W_i, i = 1, 2, 3$ with $W_i : [0, \infty[\to [0, \infty[$, $W_i(0) = 0$, W_i strictly increasing, and $W_1(t) \to \infty$, as $t \to \infty$. Further assume that there exists a Liapunov function $V : E \to \Re$ such that for some $M > 0$, when x is an integral solution of Eq.(1.1) with $|x(t)| \ge M$, then

$$W_1(|x(t)|) \le V(x(t)) \le W_2(|x(t)|), \text{ and}$$
$$\tfrac{d}{dt} V(x(t)) \le -W_3(|x(t)|) \text{ or } V(x(t)) - V(x(0)) \le -\int_0^t W_3(|x(s)|) ds.$$

Then integral solutions of Eq.(1.1) are bounded and ultimate bounded.

4   AN EXAMPLE.

In the following, we apply our results here to the partial functional differential equation

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + f(t, x, u_t(\cdot, x)), \ u(t, 0) = u(t, 1) = 0, \ t \ge 0, \ x \in [0, 1], \qquad (4.1)$$

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where $u_t(\cdot, x)$ (not a partial derivative) is defined by $u_t(\cdot, x)(\theta) = u(t + \theta, x)$, $\theta \in [-r, 0]$, $r > 0$ is a constant.

We study Eq.(4.1) in $E = C[0, 1]$, the space of all continuous functions on $[0, 1]$ with the super-norm, and define

$$Au = u'', \quad D(A) = \{u \in C^2[0, 1] : u(0) = u(1) = 0\}. \tag{4.2}$$

Then the closure of $D(A)$ is

$$\overline{D(A)} = C_0[0, 1] = \{u \in C[0, 1] : u(0) = u(1) = 0\} \neq C[0, 1] = E, \tag{4.3}$$

thus A is not densely defined on E.

Now, (10.2) in [4] is true, and it implies (1.1) in [4]. Thus Hille-Yosida condition (H_1) in our paper is satisfied.

Let E_0 be the closure of $D(A)$, that is,

$$E_0 = \overline{D(A)} = \{u \in C[0, 1] : u(0) = u(1) = 0\},$$

and define the part of A as

$$A_0 u = Au = u'', \tag{4.4}$$

on the domain

$$\begin{aligned} D(A_0) &= \{u \in D(A) : Au \in E_0\} \\ &= \{u \in C^2[0, 1] : u(0) = u(1) = u''(0) = u''(1) = 0\}. \end{aligned} \tag{4.5}$$

Lemma 4.1. A_0 generates a compact semigroup on $E_0 = \overline{D(A)}$.

Proof: Let $f \in E_0$ and consider $(\lambda - A_0)u = f$ for $\lambda \notin (-\infty, 0]$, that is,

$$\lambda u(t) - u''(t) = f(t), \quad u(0) = u(1) = u''(0) = u''(1) = 0. \tag{4.6}$$

From [5], $\lambda u(t) - u''(t) = f(t)$, $u(0) = u(1) = 0$ has a unique solution w (given on page 372 of [5]). Now, $w \in E_0$ and $f \in E_0$, so $w'' = \lambda w - f \in E_0$. Thus w is also a unique solution of Eq.(4.6). Therefore $(\lambda - A_0)^{-1}$ is well defined on E_0 for $\lambda \notin (-\infty, 0]$.

Next, similar to the argument in [5], we have

$$\|(\lambda - A_0)^{-1}\|_{B(E_0)} \leq \frac{1}{|\lambda| \cos(q/2)}, \tag{4.7}$$

where $q = \arg \lambda$ and $B(E_0)$ is the Banach space of all bounded linear operators on E_0. Now from [4] or [10], A_0 generates an analytic semigroup on E_0. Then similar to [10], the analytic semigroup is continuous in the uniform operator topology for $t > 0$; and

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$(\lambda - A_0)^{-1}$, $\lambda \notin (-\infty, 0]$, maps bounded set in E_0 into bounded set in E_0 with a uniform bound on their first derivatives. It now follows from Arzela-Ascoli's theorem that $(\lambda - A_0)^{-1}$, $\lambda \notin (-\infty, 0]$, is a compact operator. Therefore the semigroup in compact according to [10]. □

For this example, conditions (H_1)–(H_4) can be met, so that results here can be applied. Details are left to the reader.

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