MATHEMATICAL
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# Nondensely Defined Evolution Equations with Nonlocal Conditions 

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#### Abstract

In this note, we establish some results concerning the existence and regularity of "integral solutions" for some nondensely defined evolution equations with nonlocal condition, where the linear part satisfies the Hille-Yosida condition. They extend the results of densely defined evolution equations. © 2002 Elsevier Science Ltd. All rights reserved.


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## 1. INTRODUCTION

In this work, we are concerned with the following nonlocal evolution equation:

$$
\begin{align*}
\frac{d u(t)}{d t} & =A u(t)+F(t, u(t)), \quad t \in[0, T],  \tag{1}\\
u(0)+g(u) & =u_{0},
\end{align*}
$$

where $A: D(A) \subset E \rightarrow E$, is a nondensely defined closed linear operator on a Banach space $E$, $F:[0, T] \times E \rightarrow E$ is continuous, and $g: C([0, T] ; E) \rightarrow E$ is continuous, where $C([0, T] ; E)$ is the space of continuous functions with the uniform norm topology.

As indicated in $[1-4]$ and the references therein, the nonlocal condition $u(0)+g(u)=u_{0}$ can be applied in physics with better effect than the classical Cauchy problem $u(0)=u_{0}$. For example, in [4], the author used

$$
\begin{equation*}
g(u)=\sum_{i=1}^{p} c_{i} u\left(t_{i}\right), \tag{2}
\end{equation*}
$$

[^0]where $c_{i}, i=1, \ldots, p$, are given constants and $0<t_{1}<t_{2}<\cdots<t_{p} \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, equation (2) allows the additional measurements at $t_{i}, i=1,2, \ldots, p$.

When operator $A$ generates a $C_{0}$ semigroup, or equivalently, when a closed linear operator $A$ satisfies
(i) $\overline{D(A)}=E,(D$ means domain),
(ii) the Hille-Yosida condition that is, there exist $M \geq 0$ and $\tau \in \Re$ such that $] \tau, \infty[\subset$ $\rho(A), \sup \left\{(\lambda-\tau)^{n}\left|(\lambda I-A)^{-n}\right|: \lambda>\tau, n \in N\right\} \leq M$,
where $\rho(A)$ is the resolvent set of $A$ and $I$ is the identity operator, then equation (1) with nonlocal conditions have been studied extensively. Existence, uniqueness, and regularity, among other things, are derived. See [1-7]. See also [8] for a study of integrodifferential equations with nonlocal conditions,

$$
\begin{gather*}
\frac{d u(t)}{d t}=A\left[u(t)+\int_{0}^{t} H(t-s) u(s) d s\right]+f(t, u(t)), \quad 0 \leq t \leq T  \tag{3}\\
u(0)+h\left(t_{1}, \ldots, t_{p}, u\left(t_{1}\right), \ldots u\left(t_{p}\right)\right)=u_{0}
\end{gather*}
$$

where $A$ generates a $C_{0}$ semigroup on a Banach space and $h:[0, T]^{p} \times E^{p} \rightarrow E$ is continuous and $H(t)$ is a bounded linear operator for $t \in[0, T]$.

However, as indicated in [9], we sometimes need to deal with the nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,1]$ and consider $A=\frac{\partial^{2}}{\partial^{2} x}$ in $C([0,1], \Re)$ in order to measure the solutions in the sup-norm, then domain

$$
D(A)=\left\{\phi \in C^{2}([0,1], \Re): \phi(0)=\phi(1)=0\right\}
$$

is not dense in $C([0,1], \Re)$ with the sup-norm. See [9] for more examples and remarks concerning the nondensely defined operators.

Therefore, our purpose here is to extend the results of densely defined evolution equations with nonlocal conditions to nondensely defined evolution equations with nonlocal conditions. We first use the fixed-point methods to derive the existence and uniqueness of "integral solutions" (when the operator $A$ is nondensely defined), then we verify that integral solutions are "strict solutions" if additional conditions are assumed. In doing so, we apply the two approaches that are commonly used in recent publications: the contraction mapping principle and Leray-Schauder's alternative. We remark that Leray-Schauder's alternative requires some compactness conditions; thus, to use it, some compactness conditions must be imposed on the function $g$ in equation (1). In this regard, we point out that the proof of the main result in [7] is not complete and additional compactness conditions on $g$ should be assumed there for the proof to go through. See $[6,10]$ for similar remarks.

This paper is organized as follows. In Section 2, we state some facts about integrated semigroups and integral solutions that will be used later. In Section 3, we prove the existence, uniqueness, and regularity of integral solutions for equation (1) when $A$ is not necessarily densely defined but satisfies the Hille-Yosida condition. We also give an optimal result about the existence of integral solutions when $A$ is a sectorial operator. The remaining section is devoted to an application.

## 2. PRELIMINARIES AND BASIC RESULTS

Let us introduce some notions which will be used in this paper.
Definition 1. (See [11].) Let $E$ be a real Banach space. An integrated semigroup is a family $(S(t))_{t \geq 0}$ of bounded linear operators $S(t)$ on $E$ with the following properties:
(i) $S(0)=0$;
(ii) $t \rightarrow S(t)$ is strongly continuous;
(iii) $S(s) S(t)=\int_{0}^{s}(S(t+r)-S(r)) d r$, for all $t, s \geq 0$.

Definition 2. (See [12].) An operator $A$ is called a generator of an integrated semigroup if there exists $\omega \in \Re$ such that $] \omega, \infty[\subset \rho(A)(\rho(A)$, is the resolvent set of $A)$ and there exists a strongly continuous exponentially bounded family $(S(t))_{t \geq 0}$ of bounded operators such that $S(0)=0$ and $(\lambda I-A)^{-1}=\lambda \int_{0}^{\infty} e^{-\lambda t} S(t) d t$ exists for all $\lambda$ with $\lambda>\omega$.
Proposition 1. (See [11].) Let $A$ be the generator of an integrated semigroup $(S(t))_{t \geq 0}$. Then for all $x \in E$ and $t \geq 0$,

$$
\int_{0}^{t} S(s) x d s \in D(A) \quad \text { and } \quad S(t) x=A \int_{0}^{t} S(s) x d s+t x
$$

## Definition 3. (See [12].)

(i) An integrated semigroup $(S(t))_{t \geq 0}$ is called locally Lipschitz continuous if, for all $\tau>0$. there exists a constant $L$ such that

$$
|S(t)-S(s)| \leq L|t-s|, \quad t, s \in[0, \tau]
$$

(ii) An integrated semigroup $(S(t))_{t \geq 0}$ is called nondegenerate if $S(t) x=0$, for all $t \geq 0$ implies that $x=0$.

Definition 4. We say that a linear operator A satisfies the "Hille-Yosida condition" if there exist $M \geq 0$ and $\omega \in \Re$ such that $] \omega, \infty[\subset \rho(A)$ and

$$
\sup \left\{(\lambda-\omega)^{n}\left|(\lambda I-A)^{-n}\right|: n \in N, \lambda>\omega\right\} \leq M
$$

Theorem 2. (See [12].) The following assertions are equivalent:
(i) A is the generator of a nondegenerate, locally Lipschitz continuous integrated semigroup;
(ii) A satisfies the Hille-Yosida condition.

If $A$ is the generator of an integrated semigroup $(S(t))_{t \geq 0}$ which is locally Lipschitz, then from [11], $S()$.$x is continuously differentiable if and only if x \in \overline{D(A)}$ and $\left(S^{\prime}(t)\right)_{t \geq 0}$ is a $C_{0}$ semigroup on $\overline{D(A)}$. Here and hereafter, we assume that
$\left(\mathrm{H}_{1}\right) A$ satisfies the Hille-Yosida condition.
Let $(S(t))_{t \geq 0}$, be the integrated semigroup generated by $A$, then one has the following.
Theorem 3: (See $[11,12]$.) Let $f:[0, T] \rightarrow E$ be a continuous function. Then for $y_{0} \in \overline{D(A)}$, there exists a unique continuous function $y:[0, T] \rightarrow E$ such that
(i) $\int_{0}^{t} y(s) d s \in D(A), t \in[0, T]$,
(ii) $y(t)=y_{0}+A \int_{0}^{t} y(s) d s+\int_{0}^{t} f(s) d s, t \in[0, T]$,
(iii) $|y(t)| \leq M e^{\omega t}\left(\left|y_{0}\right|+\int_{0}^{t} e^{-\omega s}|f(s)| d s\right), t \in[0, T]$.

Moreover, $y$ satisfies the following variation of constant formula:

$$
\begin{equation*}
y(t)=S^{\prime}(t) y_{0}+\frac{d}{d t} \int_{0}^{t} S(t-s) f(s) d s, \quad t \geq 0 \tag{4}
\end{equation*}
$$

Let $B_{\lambda}=\lambda R(\lambda, A)$, then for all $x \in \overline{D(A)}, B_{\lambda} x \rightarrow x$ as $\lambda \rightarrow \infty$. As a consequence, if $y$ satisfies (4), then

$$
\begin{equation*}
y(t)=S^{\prime}(t) y_{0}+\lim _{\lambda \rightarrow \infty} \int_{0}^{t} S^{\prime}(t-s) B_{\lambda} f(s) d s, \quad t \geq 0 \tag{5}
\end{equation*}
$$

## 3. EXISTENCE AND REGULARITY OF SOLUTIONS

Now, we study equation (1) with nonlocal conditions.
Definition 5. We say that $u \in C([0, T] ; E)$ is an integral solution of equation (1) if the following assertions are true:
(i) $\int_{0}^{t} u(s) d s \in D(A), t \in[0, T]$;
(ii) $u(t)=u_{0}-g(u)+A \int_{0}^{t} u(s) d s+\int_{0}^{t} F(s, u(s)) d s, t \in[0, T]$.

DEFINITION 6. We say that $u \in C([0, T] ; E)$ is a strict solution of equation (1) if $u \in C^{1}([0 . T!$ : E) $\cap C([0, T] ; D(A))$ and $u$ satisfies equation (1).

## Remark 1.

(A) If $u$ is an integral solution of equation (1), then for all $t \in[0, T], u(t) \in \overline{D(A)}$. In particular, $u(0) \in \overline{D(A)}$. If the integral solution $u$ exists then by Theorem 3 , it is given by

$$
u(t)=S^{\prime}(t)\left[u_{0}-g(u)\right]+\frac{d}{d t} \int_{0}^{t} S(t-s) F(s, u(s)) d s, \quad \text { for } t \in[0, T]
$$

(B) If $u$ is an integral solution of equation (1) such that $u \in C^{1}([0, T] ; E)$ or $u \in C([0, T]$; $D(A)$ ), then $u$ is a strict solution of equation (1).

### 3.1. Integral and Strict Solutions

In the sequel, we assume the following.
$\left(\mathrm{H}_{2}\right)$ There exists a positive constant $a$ such that

$$
|F(t, x)-F(t, y)| \leq a|x-y|, \quad t \in[0, T], \quad x, y \in E .
$$

$\left(\mathrm{H}_{3}\right) g: C([0, T] ; E) \rightarrow \overline{D(A)}$ and there exists a positive constant $b$ such that

$$
|g(u)-g(v)| \leq b\|u-v\|_{C}, \quad u, v \in C([0, T] ; E)
$$

Now, let $r>0$ and define the following constants,

$$
N=\sup _{\|u\|_{C} \leq r}|g(u)| \quad \text { and } \quad c=\sup _{t \in[0, T]}|F(t, 0)| .
$$

Theorem 4. Assume that Assumptions $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{3}\right)$ hold. Let $r, T>0$ be chosen such that

$$
\begin{equation*}
M e^{\omega T}\left[\left|u_{0}\right|+N+(a r+c) \frac{1}{\omega}\left(1-e^{-\omega T}\right)\right] \leq r \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
M e^{\omega T}\left[b+\frac{a}{\omega}\left(1-e^{-\omega T}\right)\right]<1 \tag{7}
\end{equation*}
$$

where $M$ and $\omega$ are from the Hille-Yosida condition. Let $u_{0} \in \overline{D(A)}$, then equation (1) has a unique integral solution $u$ on $[0, T]$.
Proof. Let $u_{0} \in \overline{D(A)}$ and let $C_{r}=\left\{u \in C([0, T] ; E):\|u\|_{C} \leq r\right\}$. For $u \in C([0, T] ; E)$, define $K u$ by

$$
K u(t)=S^{\prime}(t)\left[u_{0}-g(u)\right]+\frac{d}{d t} \int_{0}^{t} S(t-s) F(s, u(s)) d s, \quad t \in[0, T]
$$

We will apply the contraction mapping principle to show that $K$ has a unique fixed point. For $u \in C_{r}$, we can use the results in Theorem 3 and condition (6) to obtain

$$
\begin{align*}
|K u(t)| & \leq M e^{\omega t}\left[\left|u_{0}\right|+N+\int_{0}^{t} e^{-\omega s}|F(s, u(s))| d s\right] \\
& \leq M e^{\omega t}\left[\left|u_{0}\right|+N+\int_{0}^{t} e^{-\omega s}|F(s, u(s))-F(s, 0)+F(s, 0)| d s\right] \\
& \leq M e^{\omega t}\left[\left|u_{0}\right|+N+\int_{0}^{t} e^{-\omega s} a|u(s)| d s+\int_{0}^{t} e^{-\omega s}|F(s, 0)| d s\right]  \tag{8}\\
& \leq M e^{\omega T}\left[\left|u_{0}\right|+N+(a r+c) \int_{0}^{T} e^{-\omega s} d s\right] \\
& \leq M e^{\omega T}\left[\left|u_{0}\right|+N+(a r+c) \frac{1}{\omega}\left(1-e^{-\omega T}\right)\right] \\
& \leq r
\end{align*}
$$

hence, we get $K C_{r} \subset C_{r}$. Next, let $u, v \in C([0, T] ; E)$, then

$$
\begin{align*}
|K u-K v| & \leq\left|S^{\prime}(t)[g(v)-g(u)]+\frac{d}{d t} \int_{0}^{t} S(t-s)[F(s, u(s))-F(s, v(s))] d s\right| \\
& \leq M e^{\omega t}\left[b\|u-v\|_{C}+\int_{0}^{t} e^{-\omega s}|F(s, u(s))-F(s, v(s))| d s\right]  \tag{9}\\
& \leq M e^{\omega T}\left[b+\frac{a}{\omega}\left(1-e^{-\omega T}\right)\right]\|u-v\|_{C}
\end{align*}
$$

Consequently, by applying condition (7) and the contraction mapping principle, $K$ has a unique fixed point. Then by Theorem 3, we see that equation (1) has a unique integral solution on $[0, T]$.

To obtain the regularity of the integral solution $u$, we assume the following.
$\left(\mathrm{H}_{4}\right) F$ is continuously differentiable.
Theorem 5. Assume that Assumptions $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{4}\right)$ hold. Let $r, T>0$ be chosen as in Theorem 4 and let $u$ be the unique integral solution of equation (1) guaranteed by Theorem 4. In addition, suppose that $u_{0}-g(u) \in D(A)$ and $A\left[u_{0}-g(u)\right]+F\left(0, u_{0}-g(u)\right) \in \overline{D(A)}$. Then $u$ is a strict solution of equation (1).
Proof. Let $u$ be the unique integral solution of equation (1) guaranteed by Theorem 4. Consider the equation

$$
\begin{align*}
\frac{d v(t)}{d t} & =A v(t)+D_{1} F(t, u(t))+D_{2} F(t, u(t)) v(t), \quad t \in[0, T],  \tag{10}\\
v(0) & =A u(0)+F(0, u(0)),
\end{align*}
$$

where $D_{1}$ and $D_{2}$ are the partial derivatives to the first and second variables, respectively. Then by the contraction mapping principle we can prove that equation (10) has an integral solution $v$ on $[0, T]$ which is given by

$$
\begin{equation*}
v(t)=S^{\prime}(t)[A u(0)+F(0, u(0))]+\frac{d}{d t} \int_{0}^{t} S(t-s)\left[D_{1} F(s, u(s))+D_{2} F(s, u(s)) v(s)\right] d s \tag{11}
\end{equation*}
$$

On the other hand, we have,

$$
\begin{equation*}
F(s+h, u(s))-F(s, u(s))=D_{1} F(s, u(s)) h+w_{1}(s, h) \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
F(s, u(s+h))-F(s, u(s))=D_{2} F(s, u(s))[u(s+h)-u(s)]+w_{2}(s, h), \tag{13}
\end{equation*}
$$

where $w_{i}(s, h) / h \rightarrow 0$ as $h \rightarrow 0$ uniformly for $s \in[0, T]$. Define

$$
v_{h}(t)=\frac{u(t+h)-u(t)}{h}-v(t), \quad t \in[0, T] .
$$

Then

$$
\begin{align*}
& u(t+h)-u(t) \\
&= {\left[S^{\prime}(t+h)-S^{\prime}(t)\right] u(0)+\frac{d}{d t} \int_{t}^{t+h} S(s) F(t+h-s, u(t+h-s)) d s } \\
&+\frac{d}{d t} \int_{0}^{t} S(s) F(t+h-s, u(t+h-s)) d s-\frac{d}{d t} \int_{0}^{t} S(s) F(t-s, u(t-s)) d s \\
&= {\left[S^{\prime}(t+h)-S^{\prime}(t)\right] u(0)+\frac{d}{d t} \int_{0}^{h} S(t+h-s) F(s, u(s)) d s }  \tag{14}\\
& \frac{d}{d t} \int_{0}^{t} S(s)[F(t+h-s, u(t+h-s))-F(t-s, u(t+h-s))] d s \\
&+\frac{d}{d t} \int_{0}^{t} S(s)[F(t-s, u(t+h-s))-F(t-s, u(t-s))] d s .
\end{align*}
$$

Then by (12),(13), one has

$$
\begin{align*}
u(t+h)-u(t)= & {\left[S^{\prime}(t+h)-S^{\prime}(t)\right] u(0)+\frac{d}{d t} \int_{0}^{h} S(t+h-s) F(s, u(s)) d s } \\
& +h \frac{d}{d t} \int_{0}^{t} S(t-s) D_{1} F(s, u(s+h)) d s+\frac{d}{d t} \int_{0}^{t} S(t-s) w_{1}(s, h) d s  \tag{15}\\
& +\frac{d}{d t} \int_{0}^{t} S(t-s) D_{2} F(s, u(s))[u(s+h)-u(s)] d s+\frac{d}{d t} \int_{0}^{t} S(t-s) w_{2}(s, h) d s
\end{align*}
$$

Thus, we have

$$
\begin{align*}
v_{h}(t)= & \frac{S^{\prime}(t+h)-S^{\prime}(t)}{h} u(0)-S^{\prime}(t)[A u(0)+F(0, u(0))] \\
& +\frac{d}{d t} \int_{0}^{t} S(t-s)\left[D_{1} F(s, u(s+h))-D_{1} F(s, u(s))\right] d s \\
& +\frac{1}{h} \frac{d}{d t} \int_{0}^{t} S(t-s)\left[w_{1}(s, h)+w_{2}(s, h)\right] d s  \tag{16}\\
& +\frac{1}{h} \frac{d}{d t} \int_{0}^{h} S(t+h-s) F(s, u(s)) d s+\frac{d}{d t} \int_{0}^{t} S(t-s) D_{2} F(s, u(s)) v_{h}(s) d s
\end{align*}
$$

On the other hand, by Proposition 1, we have

$$
\frac{S^{\prime}(t+h)-S^{\prime}(t)}{h} u(0)=\frac{S(t+h)-S(t)}{h} A u(0)
$$

Therefore, from

$$
\frac{S^{\prime}(t+h)-S^{\prime}(t)}{h} u(0)=\frac{S(t+h)-S(t)}{h}[A u(0)+F(0, u(0))]-\frac{S(t+h)-S(t)}{h} F(0, u(0))
$$

and

$$
\begin{aligned}
\frac{S(t+h)-S(t)}{h} F(0, u(0)) & =\frac{1}{h} \frac{d}{d t} \int_{t}^{t+h} S(s) F(0, u(0)) d s \\
& =\frac{1}{h} \frac{d}{d t} \int_{0}^{h} S(t+h-s) F(0, u(0)) d s
\end{aligned}
$$

we conclude that (16) becomes

$$
\begin{align*}
v_{h}(t)= & \left\{\frac{S(t+h)-S(t)}{h}[A u(0)+F(0, u(0))]-S^{\prime}(t)[A u(0)+F(0, u(0))]\right\} \\
& +\frac{d}{d t} \int_{0}^{t} S(t-s)\left[D_{1} F(s, u(s+h))-D_{1} F(s, u(s))\right] d s \\
& +\frac{1}{h} \frac{d}{d t} \int_{0}^{t} S(t-s)\left[w_{1}(s, h)+w_{2}(s, h)\right] d s  \tag{17}\\
& +\frac{1}{h} \frac{d}{d t} \int_{0}^{h} S(t+h-s)[F(s, u(s))-F(0, u(0))] d s \\
& +\frac{d}{d t} \int_{0}^{t} S(t-s) D_{2} F(s, u(s)) v_{h}(s) d s
\end{align*}
$$

It is now clear that the first four terms on the right-hand side of (17) go to zero as $h \rightarrow 0$. Consequently, there exists a positive constant $d$ such that

$$
\left|v_{h}(t)\right| \leq \varepsilon(h)+d \int_{0}^{t}\left|v_{h}(s)\right| d s
$$

with $\varepsilon(h) \rightarrow 0$ as $h \rightarrow 0$. By Gronwall's lemma, $v_{h}(t) \rightarrow 0, h \rightarrow 0$. Thus, $u$ is continuously differentiable on $[0, T]$. Hence, $u$ is a strict solution of equation (1).

### 3.2. The General Case

Here, we will prove the existence of integral solutions under the conditions given in [7], which are weaker than the Lipschitz condition assumed in Theorem 5. Among those conditions, we note that $F$ is not required to be continuous. Now, we list the conditions in [7].
$\left(H_{5}\right)$ For each $t \in[0, T]$, the function $F(t,$.$) is continuous; and for each x$, the function $F(., x)$ is strongly measurable.
$\left(\mathrm{H}_{6}\right)$ The operator $S^{\prime}(t)$ is compact in $\overline{D(A)}$ whenever $t>0$.
$\left(\mathrm{H}_{7}\right)$ There exists a continuous function $m:[0, T] \rightarrow R^{+}$and a continuous nondecreasing function $\Omega:[0, \infty[\rightarrow[0, \infty[$ such that

$$
|F(t, x)| \leq m(t) \Omega(|x|), \quad t \in[0, T], \quad x \in E .
$$

$\left(\mathrm{H}_{8}\right) g: C([0, T] ; \overline{D(A)}) \rightarrow \overline{D(A)}$ is compact (i.e., continuous and takes a bounded set into a compact set) and there exists $G>0$ such that $|g(u)| \leq G$, for all $u$.

We also need the following well-known Leray-Schauder lemma.
Lemma 6. (See [7].) Let $V$ be a convex subset of a normed linear space $X$ and assume $0 \in V$. Let $K: V \rightarrow V$ be compact (i.e., continuous and takes a bounded set into a compact set). Let

$$
\xi(K)=\{x \in V: x=\lambda K x, \text { for some } 0<\lambda<1\} .
$$

Then either
(i) $\xi(K)$ is unbounded, or
(ii) $K$ has a fixed point.

Under these conditions, we are going to prove the existence of integral solutions for equation (1) using the idea similar to the one in [7]. However, we point out that additional compactness conditions should be assumed in order to use the Leray-Schauder lemma.

Theorem 7. Assume that Assumptions $\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{5}\right)-\left(\mathrm{H}_{8}\right)$ hold. If $u_{0} \in \overline{D(A)}$, and

$$
\begin{equation*}
\int_{0}^{T} \max (\omega, M m(s)) d s<\int_{M\left(\left|u_{0}\right|+G\right)}^{\infty} \frac{d s}{s+\Omega(s)}, \tag{18}
\end{equation*}
$$

where $M$ and $\omega$ are from the Hille-Yosida condition, then equation (1) has at least one integral solution on $[0, T]$.
Proof. Let $K: C([0, T] ; \overline{D(A)}) \rightarrow C([0, T] ; \overline{D(A)})$ be defined by

$$
K u(t)=S^{\prime}(t)\left[u_{0}-g(u)\right]+\frac{d}{d t} \int_{0}^{t} S(t-s) F(s, u(s)) d s, \quad t \in[0, T]
$$

We claim that $\xi(K)$ is bounded. In fact, for $u \in \xi(K)$, there exists $\lambda \in(0,1)$ such that $u=\lambda K u$; that is

$$
u(t)=\lambda S^{\prime}(t)\left[u_{0}-g(u)\right]+\lambda \frac{d}{d t} \int_{0}^{t} S(t-s) F(s, u(s)) d s, \quad t \in[0, T]
$$

Using Assumptions $\left(\mathrm{H}_{7}\right),\left(\mathrm{H}_{8}\right)$, we get

$$
\begin{equation*}
e^{-\omega t}|u(t)| \leq M\left[\left|u_{0}\right|+G+\int_{0}^{t} e^{-\omega s} m(s) \Omega(|u(s)|) d s\right] . \tag{19}
\end{equation*}
$$

Let $z(t)$ denote the right-hand side of (19), then

$$
z^{\prime}(t)=M e^{-\omega t} m(t) \Omega(|u(t)|), \quad \text { for } t \in[0, T] \quad \text { and } \quad z(0)=M\left(\left|u_{0}\right|+G\right) .
$$

From (19), we have $|u(t)| \leq e^{\omega t} z(t)$. Then

$$
z^{\prime}(t) \leq M e^{-\omega t} m(t) \Omega\left(e^{\omega t} z(t)\right), \quad t \in[0, T] .
$$

Accordingly, we have

$$
\left(e^{\omega t} z(t)\right)^{\prime} \leq \max \{\omega, M m(t)\}\left(e^{\omega t} z(t)+\Omega\left(e^{\omega t} z(t)\right)\right), \quad t \in[0, T],
$$

which implies, from assumption (18), that

$$
\int_{\left.M\left(\left|u_{0}\right|+G\right)\right)}^{e^{\omega t} z(t)} \frac{d s}{s+\Omega(s)} \leq \int_{0}^{T} \max (\omega, M m(s)) d s<\int_{M\left(\left|u_{v}\right|+G\right)}^{\infty} \frac{d s}{s+\Omega(s)}, \quad t \in[0, T] .
$$

Now, we deduce that there exists a positive constant $\alpha$ which depends on $T$ and the functions $m$ and $\Omega$ such that $|u(t)| \leq \alpha$ for all $u \in \xi(K)$, which implies that $\xi(H)$ is bounded.
It remains to prove that $K$ is compact. Let $\left\{y_{n}\right\}$ be a sequence in $C=C([0, T] ; \overline{D(A)})$ with $\lim _{n \rightarrow \infty} y_{n}=y$ in $C$. By the continuity of $F$ with respect to the second argument, we deduce that for each $s \in[0, T], F\left(s, y_{n}(s)\right)$ converges to $F(s, y(s))$ in $E$; and we have

$$
\left|K y_{n}-K y\right| \leq M e^{\omega T}\left[\left|g\left(y_{n}\right)-g(y)\right|+\int_{0}^{T} e^{-\omega s}\left|F\left(s, y_{n}(s)\right)-F(s, y(s))\right| d s\right]
$$

The sequence $\left\{y_{n}\right\}$ is bounded in $C$, then by Assumption ( $\mathrm{H}_{7}$ ), and using Lebesgue dominated convergence theorem and the continuity of $g$, we obtain that

$$
\lim _{n \rightarrow \infty} K y_{n}=K y, \quad \text { in } C,
$$

which implies that the mapping $K$ is continuous on $C$.

Next, we use Ascoli-Arzela's theorem to prove that $K$ maps every bounded set into a compact set. Let $B$ be a bounded set of $C$ and let $t \in[0, T]$ be fixed, then we need to prove that $\{K y(t): y \in B\}$ is relatively compact in $E$. If $t=0$, then $\{K y(0): y \in B\}=\left\{u_{0}-g(y): y \in B\right\}$ is relatively compact since we assumed that $g$ is compact. If $t \in(0, T]$, choose $\varepsilon$ such that $0<\varepsilon<t$. Then

$$
\begin{aligned}
K y(t)= & S^{\prime}(t)\left[u_{0}-g(y)\right]+\lim _{\lambda \rightarrow \infty} \int_{0}^{t} S^{\prime}(t-s) B_{\lambda} F(s, y(s)) d s \\
= & S^{\prime}(t)\left[u_{0}-g(y)\right]+S^{\prime}(\varepsilon) \lim _{\lambda \rightarrow \infty} \int_{0}^{t-\varepsilon} S^{\prime}(t-\varepsilon-s) B_{\lambda} F(s, y(s)) d s \\
& +\lim _{\lambda \rightarrow \infty} \int_{t-\varepsilon}^{t} S^{\prime}(t-s) B_{\lambda} F(s, y(s)) d s
\end{aligned}
$$

Since $S^{\prime}(\varepsilon)$ is compact, we deduce that there exists a compact set $D_{1}$ such that

$$
S^{\prime}(\varepsilon) \lim _{\lambda \rightarrow \infty} \int_{0}^{t-\varepsilon} S^{\prime}(t-\varepsilon-s) B_{\lambda} F(s, y(s)) d s \in D_{1}, \quad \text { for } y \in B
$$

Furthermore, by Assumption $\left(\mathrm{H}_{7}\right)$, there exists a positive constant $b_{1}$ such that

$$
\left|\lim _{\lambda \rightarrow \infty} \int_{t-\varepsilon}^{t} S^{\prime}(t-s) B_{\lambda} F(s, y(s)) d s\right| \leq b_{1} \varepsilon, \quad \text { for } y \in B
$$

Moreover, $g$ is compact, it follows that $\left\{S^{\prime}(t)\left[u_{0}-g(y)\right]: y \in B\right\}$ is relatively compact. We conclude that $\{K y(t): y \in B\}$ is totally bounded, and therefore, it is relatively compact in $E$.

Finally, let us show that $K B$ is equicontinuous. For every $0 \leq t_{0}<t \leq T$ and $y \in B$,

$$
\begin{aligned}
K y(t)-K y\left(t_{0}\right)= & {\left[S^{\prime}(t)-S^{\prime}\left(t_{0}\right)\right]\left[u_{0}-g(y)\right] } \\
& +\lim _{\lambda \rightarrow \infty} \int_{0}^{t_{0}}\left[S^{\prime}(t-s)-S^{\prime}\left(t_{0}-s\right)\right] B_{\lambda} F(s, y(s)) d s \\
& +\lim _{\lambda \rightarrow \infty} \int_{t_{0}}^{t} S^{\prime}(t-s) B_{\lambda} F(s, y(s)) d s
\end{aligned}
$$

Next, we have

$$
\sup _{y \in B}\left|\int_{t_{0}}^{t} S^{\prime}(t-s) B_{\lambda} F(s, y(s)) d s\right| \rightarrow 0, \quad t \rightarrow t_{0}
$$

and

$$
\begin{aligned}
& \lim _{\lambda \rightarrow \infty} \int_{0}^{t_{0}}\left[S^{\prime}(t-s)-S^{\prime}\left(t_{0}-s\right)\right] B_{\lambda} F(s, y(s)) d s \\
&=\left[S^{\prime}\left(t-t_{0}\right)-I\right] \lim _{\lambda \rightarrow \infty} \int_{0}^{t_{0}} S^{\prime}\left(t_{0}-s\right) B_{\lambda} F(s, y(s)) d s
\end{aligned}
$$

Since we have proved that $\{K y(t): y \in B\}$ is relatively compact for any fixed $t \in[0, T\}$, there exists a compact set $D_{2}$ such that

$$
\lim _{\lambda \rightarrow \infty} \int_{0}^{t_{0}} S^{\prime}\left(t_{0}-s\right) B_{\lambda} F(s, y(s)) d s \in D_{2}, \quad \text { for } y \in B
$$

Now, it is well known from the operator theory that

$$
\lim _{h \rightarrow 0^{+}} \sup _{z \in D_{2}}\left|\left(S^{\prime}(h)-I\right) z\right|=0 .
$$

Moreover, $\left\{u_{0}-g(y): y \in B\right\}$ is relatively compact, therefore, we obtain

$$
\begin{equation*}
\lim _{t \rightarrow t_{0}^{+}} \sup _{y \in B}\left|K y(t)-K y\left(t_{0}\right)\right|=0 \tag{20}
\end{equation*}
$$

Similarly, for $0<t_{0}$, we can prove

$$
\begin{equation*}
\lim _{t \rightarrow t_{0}^{-}} \sup _{y \in B}\left|K y(t)-K y\left(t_{0}\right)\right|=0 \tag{21}
\end{equation*}
$$

Thus, $K B$ is equicontinuous. Consequently, the mapping $K$ is compact and Lemma 6 implies that $K$ has at least one fixed point, which gives rise to an integral solution of equation (1).

### 3.3. Optimal Regularity

In this section, we study the optimal regularity for equation (1), with additional regularity conditions on the operator $A$, such as the sectorial condition. For more details about this topic. see [13]. In what follows, all the notations are taken from [13]. We suppose the following.
$\left(\mathrm{H}_{9}\right) A$ is nondensely defined and sectorial; that means there exist $\omega \in \Re, \theta \in(\pi / 2, \pi), M>0$ such that

$$
\begin{aligned}
\rho(A) \supseteq S_{\theta, \omega} & =\{\lambda \in C: \lambda \neq \omega \text { and }|\arg (\lambda-\omega)|<\theta\}, \\
|R(\lambda, A)| & \leq \frac{M}{|\lambda-\omega|}, \quad \text { for all } \lambda \in S_{\theta, \omega} .
\end{aligned}
$$

Now, for every $t>0$, define the linear operator $e^{t A}$ by

$$
e^{t A}=\frac{1}{2 \pi i} \int_{\omega+\partial \Omega_{r, v}} e^{t \lambda} R(\lambda, A) d \lambda,
$$

where $r>0, \eta \in(\pi / 2, \theta)$ and $\partial \Omega_{r, \eta}$ is the curve $\{\lambda \in C:|\arg \lambda|=\eta,|\lambda| \geq r\} \cup\{\lambda \in C$ : $|\arg \lambda| \leq \eta,|\lambda|=r\}$. Let $y_{0} \in \overline{D(A)}$, and let $y$ be the function satisfying (i)-(iii) of Theorem 3, then from [13], $y$ satisfies

$$
\begin{equation*}
y(t)=e^{t \Lambda} y_{0}+\int_{0}^{t} e^{(t s) A} f(s) d s, \quad t \in[0, T] . \tag{22}
\end{equation*}
$$

Theorem 8. Assume that assumptions of Theorems 4 or 7 are satisfied. Suppose that there exist $\eta>0$ and $\theta \in(0,1)$ such that

$$
\begin{equation*}
|F(t, x)-F(s, y)| \leq \eta\left[|t-s|^{\theta}+|x-y|\right], \quad \text { for } t, s \in(0, T] \quad \text { and } \quad x, y \in E . \tag{23}
\end{equation*}
$$

Let $u_{0} \in \overline{D(A)}$, and let $u$ be an integral solution of equation (1) guaranteed by Theorems 4 or 7 . Then $u$ is continuously differentiable in $(0, T]$ and satisfies equation (1) for $t>0$.

The proof is based on the following lemmas. Let $I$ be an interval of $\Re$, then the Banach space of Holder continuous functions, $C^{\alpha}(I, E), \alpha \in(0,1)$, is defined by

$$
C^{\alpha}(I, E)=\left\{f \in C(I, E): \sup _{t, s \in I, t \neq s} \frac{|f(t)-f(s)|}{|t-s|^{\alpha}}<\infty\right\} .
$$

Lemma 9. (See [13].) Let $f \in L^{1}(0, T ; E) \cap L^{\infty}([\varepsilon, T] ; E)$ for every $\varepsilon \in(0, T)$. If in addition $y_{0} \in \overline{D(A)}$, then $y$ in (22) belongs to $C^{\alpha}([\varepsilon, T] ; E)$ for every $\varepsilon \in(0, T)$ and $\alpha \in(0,1)$.
Lemma 10. (See [13!.) Let $\theta \in(0,1)$ and let $f \in L^{1}(0, T ; E) \cap C^{\theta}([\varepsilon, T] ; E)$ for every $\varepsilon \in(0, T)$. If in addition $y_{0} \in \overline{D(A)}$, then $y$ in (22) belongs to $C^{1}(0, T ; E)$ and satisfies $y^{\prime}(t)=A y(t)+$ $f(t), y(0)=y_{0}$, for $t \in(0, T]$.
Proof of Theorem 8. According to formula (22), $u$ is now given by

$$
u(t)=e^{t A}\left[u_{0}-g(u)\right]+\int_{0}^{t} e^{(t-s) A} F(s, u(s)) d s, \quad t \in[0, T] .
$$

By Lemma 9, the integral solution $u$ belongs to $C^{\alpha}([\varepsilon, T] ; E)$ for every $\varepsilon \in(0, T)$ and $\alpha \in(0,1)$. In particular, $u$ belongs to $C^{\theta}([\varepsilon, T] ; E)$ for cvery $\varepsilon \in(0, T)$. Using assumption (23), we get that $F(., u()$.$) belongs to C^{\theta}([\varepsilon, T] ; E)$ for every $\varepsilon \in(0, T)$, which implies, by Lemma 10 , that the integral solution $u$ is continuously differentiable and satisfies equation (1) on ( $0, T]$.

### 3.4. A Special Case

In this section, we suppose that the nonlocal condition is given by

$$
g(u)=\sum_{i=1}^{p} C_{i} u\left(t_{i}\right)
$$

where $C_{i}, i=1, \ldots, p$, is a bounded operator form $\overline{D(A)}$ into $\overline{D(A)}$, such that

$$
\left|C_{i} x-C_{i} y\right| \leq \beta_{i}|x-y|, \quad x, y \in \overline{D(A)},
$$

for some positive constant $\beta_{i}, i=1,2, \ldots, p$. We make the following assumption.
$\left(\mathrm{H}_{10}\right)$ There exist positive constants $M$ and $\mu$ such that $\left|S^{\prime}(t)\right| \leq M e^{-\mu t}$, for $t \geq 0$.
Proposition 11. Assume that Assumptions $\left(\mathrm{H}_{1}\right)$, $\left(\mathrm{H}_{2}\right)$, and $\left(\mathrm{H}_{10}\right)$ hold. If we suppose that

$$
\begin{equation*}
\beta=\mu-M a>0 \quad \text { and } \quad M \sum_{i=1}^{p} \beta_{i} e^{-\beta t_{i}}<1, \tag{24}
\end{equation*}
$$

then for all $u_{0} \in \overline{D(A)}$, equation (1) has a unique integral solution on $[0, T]$.
Proof. The proof is similar to the one in [8]. Note that condition (24) is better than those given in Theorem 4. Let $v \in \overline{D(A)}$, then by using the contraction mapping principle, we can prove the existence of a unique continuous function $u:[0, T] \rightarrow \overline{D(A)}$ such that

$$
\begin{equation*}
u(t, v)=S^{\prime}(t) v+\frac{d}{d t} \int_{0}^{t} S(t-s) F(s, u(s, v)) d s, \quad \text { for } t \in[0, T] \tag{25}
\end{equation*}
$$

Define a mapping $Q: \overline{D(A)} \rightarrow \overline{D(A)}$ by

$$
Q v=u_{0}-\sum_{i=1}^{p} C_{i} u\left(t_{i}, v\right)
$$

It is sufficient to prove that $Q$ has a unique fixed point. Let $v, v^{\prime} \in \overline{D(A)}$, then

$$
\begin{equation*}
\left|Q v-Q v^{\prime}\right|=\left|\sum_{i=1}^{p} C_{i} u\left(t_{i}, v\right)-C_{i} u\left(t_{i}, v^{\prime}\right)\right| \leq \sum_{i=1}^{p} \beta_{i}\left|u\left(t_{i}, v\right)-u\left(t_{i}, v^{\prime}\right)\right| . \tag{26}
\end{equation*}
$$

Since $u(., v)$ and $u\left(., v^{\prime}\right)$ are solutions of equation (25) with initial data $v$ and $v^{\prime}$, respectively, one has

$$
\left|u(t, v)-u\left(t, v^{\prime}\right)\right| \leq M e^{-\mu t}\left[\left|v-v^{\prime}\right|+a \int_{0}^{t} e^{\mu s}\left|u(s, v)-u\left(s, v^{\prime}\right)\right| d s\right], \quad t \in[0, T] .
$$

Using Gronwall's lemma, we get

$$
\left|u(t, v)-u\left(t, v^{\prime}\right)\right| \leq M\left|v-v^{\prime}\right| e^{-(\mu-M a) t}, \quad \text { for } t \in[0, T]
$$

which implies, from (26), that

$$
\left|Q v-Q v^{\prime}\right| \leq\left(M \sum_{i=1}^{p} \beta_{i} e^{-\beta t_{i}}\right)\left|v-v^{\prime}\right|
$$

Thus, we conclude that $Q$ has a unique fixed-point $v$ such that

$$
v=u_{0}-\sum_{i=1}^{p} C_{i} u\left(t_{i}, v\right) .
$$

Therefore, (25) becomes

$$
u(t, v)=S^{\prime}(t)\left[u_{0}-\sum_{i=1}^{p} C_{i} u\left(t_{i}, v\right)\right]+\frac{d}{d t} \int_{0}^{t} S(t-s) F(s, u(s, v)) d s, \quad t \in[0, T]
$$

Accordingly, $u$ is the unique integral solution of equation (1) on $[0, T]$.

## 4. AN APPLICATION

To apply the previous result, we consider the following partial equation:

$$
\begin{array}{rlrl}
\frac{\partial}{\partial t} v(t, x) & =\Delta v(t, x)+f(t, v(t, x)), & & 0 \leq t \leq T, \\
v(t, x) & =0, & x \in \Omega  \tag{27}\\
v(0, x)+\sum_{i=1}^{p} c_{i} v\left(t_{i}, x\right) & =u_{0}(x), & & 0 \leq t \leq T, \quad x \in \partial \Omega \\
& & 0<t_{i} \leq T, \quad x \in \Omega
\end{array}
$$

where $\Omega$ is a bounded open set of $\mathbb{R}^{n}$ with regular boundary $\partial \Omega, u_{0} \in C\left(\Omega, \mathbb{R}^{n}\right), f$ is a continuous function from $[0, T] \times \bar{\Omega}$ into $\mathbb{R}$ such that for some constant $k>0$,

$$
|f(t, x)-f(t, y)| \leq k|x-y|, \quad x, y \in \bar{\Omega}, \quad t \in[0, T]
$$

and $\Delta=\sum_{k=1}^{n} \frac{\partial^{2}}{\partial x_{k}^{2}}$ and $c_{i}$ are given real numbers. Consider $E=C(\bar{\Omega})$, the Banach space of continuous functions on $\bar{\Omega}$ with values in $\mathbb{R}$. Define the linear operator $A$ in $E$ by

$$
\begin{equation*}
A z=\Delta z, \quad \text { in } D(A)=\{z \in C(\bar{\Omega}): z=0, \text { on } \partial \Omega, \Delta z \in C(\bar{\Omega})\} \tag{28}
\end{equation*}
$$

Now, we have $\overline{D(A)}=C_{0}(\bar{\Omega})=\{v \in C(\bar{\Omega}): v=0$ on $\partial \Omega\}$. It is well known from $[9 \mid$ that $A$ is sectorial, $] 0,+\infty[\subseteq \rho(A)$ and for $\lambda>0$,

$$
|R(\lambda, A)| \leq \frac{1}{\lambda}
$$

It follows that $A$ generates an integrated semigroup $(S(t))_{t \geq 0}$ and that $\left|S^{\prime}(t)\right| \leq e^{-\mu t}$ for $t \geq 0$ and some constant $\mu>0$. By the following changes:

$$
\begin{gathered}
u(t)(x)=v(t, x), \quad t \geq 0, \quad x \in \bar{\Omega} \\
F(t, u)(x)=f(t, u(x)), \quad g(u)(x)=\sum_{i=1}^{p} c_{i} u\left(t_{i}\right)(x), \quad u \in C(\bar{\Omega}), \quad x \in \bar{\Omega}
\end{gathered}
$$

equation (27) takes the abstract form (1). By Proposition 11, we have the following.
PROPOSITION 12. If we assume that $u_{0} \in C_{0}(\bar{\Omega}), k<\mu$ and $\sum_{i=1}^{p} c_{2} e^{-(\mu-k) t_{1}}<1$, then equation (27) (or equation (1)) has a unique integral solution on $[0, T]$. Moreover, if we assume that $f$ satisfies condition (23), then the integral solution of equation (27) is continuously differentiable on $(0, T]$ and satisfies equation (27) for $t>0$.

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