# INTEGRODIFFERENTIAL EQUATIONS WITH NON-AUTONOMOUS OPERATORS

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**ABSTRACT:** The result of Da Prato and Sinestrari [3] concerning the non-autonomous evolution operators of hyperbolic type for the equation

$$u'(t) = A(t)u(t) + f(t), \;\; t \in [0,T], \;\; w(0) = w_0,$$

is applied to the study of

$$u'(t) = A(t)\Big[u(t) + \int_{-\infty}^{t} G(t,s)u(s)ds\Big] + K(t)u(t) + f(t), \; t \in [0,T], \; u(s) = \varphi(s), \; s \leq 0,$$

which models linear viscoelasticity. Here $A(\cdot)$ satisfies all the requirements of Kato's semigroup approach except for the density of the common domain of $A(t)$. An application in linear viscoelasticity is given.

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## 1 INTRODUCTION.

Let us consider the following equation

$$u'(t) = A(t)u(t) + f(t), \; t \in [0,T], \;\; u(0) = u_0, \tag{1.1}$$

in a Banach space $E$. Eq. (1.1) is studied using the semigroup approach if $A(t)$ is independent of $t$. If for each fixed $t \in [0,T], A(t)$ generates a strongly continuous semigroup, then Eq. (1.1) is studied in Kato [10], Pazy [12] and Tanabe [13] using the evolution operators.

But some applications in partial differential equations involve nondensely defined operators $A(\cdot)$. For example, when seeking pointwise estimates of solutions and their derivatives in continuous functions spaces. Thus Da Prato and Sinestrari [2, 3] extended the results of Kato [10], Pazy [12] and Tanabe [13] to Eq. (1.1) with nondensely defined operators $A(\cdot)$, and proved that if $A(\cdot)$ satisfies (C4) (see Section 2, which is essentially the counterpart of the Hille − Yosida conditions in semigroup approach), and if

$$u_0 \in D, \; A(0)u_0 + f(0) \in \overline{D}, \; f \in W^{1,p}([0,T], E), \; (1 \leq p < \infty)$$

then Eq. (1.1) has a strict solution (see Definition 2.1). Here $D$ is the common domain of $A(t), t \in [0,T]$. See [1, 11, 14] for the connection of these results and the integrated semigroup theory.

We would like to apply this result to integrodifferential equations. So that, for example, equations in linear viscoelasticity can be examined.

Let us look at an equation in viscoelasticity,

$$\begin{cases} \rho(t,x)u_{tt}(t,x) + k(t,x)u_t(t,x) = \Psi_x(t,x) + h(t,x), \\ \Psi(t,x) = E(t,x)u_x(t,x) + \int_{-\infty}^t b(t,s)u_x(s,x)ds, \quad (t,x) \in [0,T] \times [0,1], \\ u(t,0) = u(t,1) = 0, \ t \in [0,T], \ \ u(s,x) = \phi(s,x), \ (s,x) \in (-\infty, 0] \times [0,1], \end{cases} \tag{1.2}$$

and see what kind of equations we are going to consider. Here the first equation is the linear momentum equation, while the second gives the constitutive relation between stress and strain. Note that

$$\begin{aligned} \Psi_x(t,x) &= \frac{\partial}{\partial x}\Big\{E(t,x)\Big[u_x(t,x) + \frac{1}{E(t,x)}\int_{-\infty}^t b(t,s)u_x(s,x)ds\Big]\Big\} \\ &= E_x(t,x)\Big[u_x(t,x) + \frac{1}{E(t,x)}\int_{-\infty}^t b(t,s)u_x(s,x)ds\Big] \\ &\quad + E(t,x)\frac{\partial}{\partial x}\Big[u_x(t,x) + \frac{1}{E(t,x)}\int_{-\infty}^t b(t,s)u_x(s,x)ds\Big]. \end{aligned}$$

So that if we let

$$w = u_t, \ v = u_x, \ ' = d/dt,$$

then we obtain

$$\begin{aligned} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' &= \begin{bmatrix} 0 & \partial_x \\ \frac{E(t)}{\rho(t)}\partial_x + \frac{E_x(t)}{\rho(t)} & 0 \end{bmatrix} \Big\{ \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \\ &\quad + \frac{1}{E(t)} \int_{-\infty}^t \begin{bmatrix} b(t,s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v(s) \\ w(s) \end{bmatrix} ds \Big\} \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & \frac{-k(t)}{\rho(t)} \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{h(t)}{\rho(t)} \end{bmatrix}, \quad t \in [0,T], \\ \big(v(s), w(s)\big) &= \big(\phi_x(s), \phi_t(s)\big), \ \ s \le 0, \end{aligned} \tag{1.3}$$

with $v(t) = v(t)(\cdot)$ and $v(t)(x) = v(t,x)$ for $x \in [0,1]$, and the same is true for $w, E, \rho, k, h$. So the equation of the form

$$\begin{cases} u'(t) = A(t)\Big[u(t) + \int_{-\infty}^t G(t,s)u(s)ds\Big] + K(t)u(t) + f(t), \ \ t \in [0,T], \\ u(s) = \varphi(s), \ s \le 0, \end{cases} \tag{1.4}$$

occurs naturally. Here $G(t,s), K(t)$ are bounded operators for $s \le t$, and $A(t)$ is an unbounded operator. Note that in this setting, we require

$$u(t) + \int_{-\infty}^t G(t,s)u(s)ds \in D(A(t)), \ (D \text{ means the domain})$$

but each term in the addition may not be in $D(A(t))$. See [7] for comments about the difference of Eq. (1.4) and other forms of integrodifferential equations.

Observe that the boundary conditions $u(t,0) = u(t,1) = 0$ in Eq. (1.2) implies $w(0) = w(1) = 0$. So if we want to obtain pointwise estimates of the solutions and their derivatives and hence consider Eq. (1.3) in continuous functions space $C[0,1] \times C[0,1]$ with the sup norm, then the leading operator in Eq. (1.3) will have domain $\{(v,w) \in C^1[0,1] \times C^1[0,1] : w(0) = w(1) = 0\}$, which is not dense in $C[0,1] \times C[0,1]$.

The method we use in Section 2 to study Eq. (1.4) is as follows: First, we define

$$w(t) = u(t) + \int_{-\infty}^{t} G(t,s)u(s)ds, \ t \geq 0,$$

and assume that $G(t,s)$ has partial derivatives so we can rewrite Eq. (1.4) as

$$\begin{aligned} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix}' &= \begin{bmatrix} K(t) & A(t) \\ K(t) + G(t,t) & A(t) \end{bmatrix} \begin{bmatrix} u(t) \\ w(t) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 0 \\ G_t(t,s) & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ w(s) \end{bmatrix} ds \\ &+ \begin{bmatrix} f(t) \\ f(t) + \int_{-\infty}^{0} G_t(t,s)\varphi(s)ds \end{bmatrix}, \quad t \in [0,T], \qquad (1.5) \\ \big(u(0), w(0)\big) &= \Big(\varphi(0), \varphi(0) + \int_{-\infty}^{0} G(0,s)\varphi(s)ds\Big), \ (u(s) = \varphi(s), s \leq 0) \end{aligned}$$

on $X = E \times E$, where $E$ is a Banach space, and prove that the leading operator in Eq. (1.5) satisfies (C4).

Next, we treat Eq. (1.5) as

$$x'(t) = Q(t)x(t) + \int_0^t H(t,s)x(s)ds + g(t), \quad t \in [0,T], \quad x(0) = x_0, \qquad (1.6)$$

on Banach space $X$, and rewrite Eq. (1.6) as an ordinary differential equation

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}' = \begin{bmatrix} Q(t) & \delta \\ H(t) & D_s \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} g(t) \\ 0 \end{bmatrix}, \ t \in [0,T], \ \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \qquad (1.7)$$

on Banach space $P = X \times F$, where $F, \delta, H(t)$ and $D_s$ will be made clear in Section 2. We will prove that the leading operator in Eq. (1.7) satisfies (C4), so that we can apply the results in Da Prato and Sinestrari [3] to Eq. (1.7).

The method of reformulating an integrodifferential equation into an ordinary differential equation in a product space has been proven to be very effective in dealing with integrodifferential equations, see [4, 5, 6, 7, 8] and the references therein for the cases when $A(\cdot)$ is a constant operator, or when $A(t)$ is densely defined.

Finally, in Section 3, we prove with some conditions on $\rho(t,x)$ and $E(t,x)$ that the leading operator in Eq. (1.3) satisfies (C4). So that we can apply the results here to study Eq. (1.3), and hence Eq. (1.2).

## 2 INTEGRODIFFERENTIAL EQUATIONS.

For the equation

$$y'(t) = P(t)y(t) + h(t), \quad t \in [0,T], \quad y(0) = y_0 \in Y_0, \qquad (2.1)$$

we first list the following conditions from Da Prato and Sinestrari [3].

(C1). For all $t \in [0,T], P(t) : Y_0 \to Y$ is a linear operator between the Banach space $(Y_0, \|\cdot\|_{Y_0})$ and $(Y, \|\cdot\|_Y)$ .

(C2). $Y_0 \subset Y$ and there exists $c > 0$ such that for all $t \in [0,T]$ and $y \in Y_0$,

$$c^{-1}\|y\|_{Y_0} \leq \|y\|_Y + \|P(t)y\|_Y \leq c\|y\|_{Y_0}.$$

(C3). There exist $\omega, M \in \Re$ (the reals) such that $(\omega, \infty) \subset \rho(P(t)), t \in [0,T]$ and for each $n \in N$ (the integers) we have

$$(\lambda - \omega)^n \|R(\lambda, t_1, \ldots, t_n)\|_{B(Y)} \leq M$$

when $t_0 \leq t_n \leq \cdots \leq t_1 \leq T$ and $\lambda > \omega$. Here we denote the resolvent set of $P(t) : Y_0 \subset Y \to Y$ by $\rho(P(t))$ and denote $B(X,Y)$ the Banach space of linear bounded operators from $X$ to $Y$, $B(X,Y) = B(Y)$ if $X = Y$, and set

$$R(\lambda, t_1, \ldots, t_n) = (\lambda - P(t_1))^{-1} \cdots (\lambda - P(t_n))^{-1}.$$

(C4). $P \in C^1([0,T], B(Y_0, Y))$ verifies (C1 – C3). More over, for each $k \in N$ (the integers) there exists $P_k \in C^4([0,T], B(Y_0, Y))$ verifying (C1 – C3) with $c, \omega, M$ independent of $k$ and such that

$$\lim_{k \to \infty} \|P - P_k\|_{C^1([0,T], B(Y_0, Y))} = 0.$$

Now we define strict solutions and state the results of Da Prato and Sinestrari [3] for reference.

**Definition 2.1.** [3] A strict solution of equation (2.1) on Banach spaces $Y_0$ and $Y$ is a function $y(\cdot) \in C^1([0,T], Y) \cap C([0,T], Y_0)$ verifying Eq. (2.1) in $[0,T]$.

**Definition 2.2.** A strict solution of Eq. (1.4) on Banach spaces $E_0$ and $E$ is a function $u(\cdot) \in C^1([0,T], E)$ verifying Eq. (1.4) in $[0,T]$ and $u(t) + \int_{-\infty}^t G(t,s)u(s)ds \in E_0$ is continuous in $t \in [0,T]$.

**Theorem 2.3.** [3] Suppose that $P(\cdot)$ satisfies (C4). Then for each $h \in W^{1,p}([0,T], Y)$ and $y_0 \in Y_0$ such that

$$P(0)y_0 + h(0) \in \overline{Y_0}, \tag{2.2}$$

equation (2.1) has a strict solution. □

Observe that by writing Eq.(1.5) in components, we have

$$\begin{aligned} u'(t) &= A(t)w(t) + K(t)u(t) + f(t), \\ w'(t) &= A(t)w(t) + K(t)u(t) + f(t) + G(t,t)u(t) + \int_{-\infty}^t G_t(t,s)u(s)ds \\ &= u'(t) + \frac{d}{dt}\int_{-\infty}^t G(t,s)u(s)ds. \end{aligned}$$

So the initial conditions in Eq.(1.5) implies

$$w(t) = u(t) + \int_{-\infty}^t G(t,s)u(s)ds, \ t \in [0,T].$$

Hence the first component in Eq.(1.5) gives rise to a solution of Eq. (1.4). Also note that it has been shown, e.g., in Desch and Schappacher [4] and Grimmer [5], that the first component of Eq. (1.7) gives rise to a solution of Eq. (1.6). So what we need to do here is to show that the leading operators in Eq.(1.5) and Eq. (1.7) satisfy (C4).

To this end, let us first prove some perturbation results.

**Lemma 2.4.** Let $P(\cdot)$ satisfy (C1 – C3) on $(Y_0, \|\cdot\|_{Y_0})$ and $(Y, \|\cdot\|_Y)$.

(i). If $P_1(\cdot) \in B(Y)$ with $\|P_1(t)\|_{B(Y)} \le \beta < 1, t \in [0,T]$, $\beta > 0$ is a constant, then $P(\cdot) + P_1(\cdot)$ satisfy (C1 – C3) on $(Y_0, \|\cdot\|_{Y_0})$ and $(Y, \|\cdot\|_Y)$.

(ii). Assume that $(Z_0, \|\cdot\|_{Z_0})$ is a Banach space in $(Y, \|\cdot\|_Y)$ and $P_2(t) \in B(Y)$ is invertible and $P_2(t) : Y_0 \to Z_0, P_2^{-1}(t) : Z_0 \to Y_0$, with $\|P_2^{-1}(t)\|_{B(Z_0,Y_0)}, \|P_2(t)\|_{B(Y_0,Z_0)}$, $\|P_2^{-1}(t)\|_{B(Y)}$ and $\|P_2(t)\|_{B(Y)} \le K, t \in [0,T]$ for a constant $K > 0$, and that for $t_2 \le t_1$, $\|P_2^{-1}(t_1)P_2(t_2)\|_{B(Y)} \le 1$. Further, assume that $P(\cdot)$ satisfy (C3) with $M = 1$. Then $P_2(\cdot)P(\cdot)P_2^{-1}(\cdot)$ satisfy (C1 – C3) on $(Z_0, \|\cdot\|_{Z_0})$ and $(Y, \|\cdot\|_Y)$.

(iii). In (ii), $M = 1$ for $P(\cdot)$ in (C3) is not required if $P_2(\cdot)$ is independent of $t$.

**Remark 1.** Conditions in perturbation results for time-dependent operators are restrictive. Fortunately, with some assumptions on $\rho(t,x), E(t,x)$ in Eq. (1.3), we are able to make suitable choices of $P_1(\cdot)$ and $P_2(\cdot)$ so as to verify that these conditions are satisfied.

**Proof.** (i). It is clear that (C1) is satisfied. Next, (C3) is true by a result in Pazy [12, page 132]. So we only need to prove that (C2) is true. Now

$$\begin{aligned}
&\|y\|_Y + \|\big[P(t) + P_1(t)\big]y\|_Y \ge \|y\|_Y + \|P(t)y\|_Y - \|P_1(t)y\|_Y \\
&\ge \|y\|_Y - \|P_1(t)\|_{B(Y)}\|y\|_Y + \|P(t)y\|_Y = (1 - \|P_1(t)\|_{B(Y)})\|y\|_Y + \|P(t)y\|_Y \\
&\ge (1-\beta)\|y\|_Y + \|P(t)y\|_Y \ge (1-\beta)\Big[\|y\|_Y + \|P(t)y\|_Y\Big] \\
&\ge (1-\beta)c^{-1}\|y\|_{Y_0},
\end{aligned}$$

where $c$ comes from (C2) for $P(\cdot)$. Similarly, we have

$$\|y\|_Y + \|\big[P(t) + P_1(t)\big]y\|_Y \le (1+\beta)\Big[\|y\|_Y + \|P(t)y\|_Y\Big] \le (1+\beta)c\|y\|_{Y_0}. \quad (2.3)$$

(Note that in proving (2.3), $\beta$ can be any positive number.)

(ii). First it is clear that (C1) is satisfied. For $z \in Z_0$,

$$\begin{aligned}
&\|z\|_Y + \|P_2(t)P(t)P_2^{-1}(t)z\|_Y = \|P_2(t)P_2^{-1}(t)z\|_Y + \|P_2(t)P(t)P_2^{-1}(t)z\|_Y \\
&\le \|P_2(t)\|_{B(Y)}\Big[\|P_2^{-1}(t)z\|_Y + \|P(t)\big(P_2^{-1}(t)z\big)\|_Y\Big] \le \|P_2(t)\|_{B(Y)}c\|P_2^{-1}(t)z\|_{Y_0} \\
&\le Kc\|P_2^{-1}(t)\|_{B(Z_0,Y_0)}\|z\|_{Z_0} \\
&\le K^2c\|z\|_{Z_0},
\end{aligned}$$

where (C2) for $P(\cdot)$ is used. Next since $\|y\|_Y \le \|P_2^{-1}(t)\|_{B(Y)}\|P_2(t)y\|_Y, y \in Y$, we have

$$\begin{aligned}
&\|z\|_Y + \|P_2(t)P(t)P_2^{-1}(t)z\|_Y = \|P_2(t)P_2^{-1}(t)z\|_Y + \|P_2(t)P(t)P_2^{-1}(t)z\|_Y \\
&\ge \frac{1}{\|P_2^{-1}(t)\|_{B(Y)}}\Big[\|P_2^{-1}(t)z\|_Y + \|P(t)\big(P_2^{-1}(t)z\big)\|_Y\Big] \ge K^{-1}c^{-1}\|P_2^{-1}(t)z\|_{Y_0} \\
&\ge K^{-2}c^{-1}\|z\|_{Z_0}, \quad (\text{since } \|z\|_{Z_0} \le \|P_2(t)\|_{B(Y_0,Z_0)}\|P_2^{-1}(t)z\|_{Y_0})
\end{aligned}$$

where, again, $c$ comes from (C2) for $P(\cdot)$. Finally, note that for $t_2 \le t_1$,

$$\begin{aligned}
&\Big(\lambda - P_2(t_1)P(t_1)P_2^{-1}(t_1)\Big)^{-1}\Big(\lambda - P_2(t_2)P(t_2)P_2^{-1}(t_2)\Big)^{-1} \\
&= \Big(P_2(t_1)\big[\lambda - P(t_1)\big]^{-1}P_2^{-1}(t_1)\Big)\Big(P_2(t_2)\big[\lambda - P(t_2)\big]^{-1}P_2^{-1}(t_2)\Big) \\
&= P_2(t_1)\big[\lambda - P(t_1)\big]^{-1}\Big(P_2^{-1}(t_1)P_2(t_2)\Big)\big[\lambda - P(t_2)\big]^{-1}P_2^{-1}(t_2).
\end{aligned}$$

So that for $P_2(\cdot)P(\cdot)P_2^{-1}(\cdot)$, we have

$$(\lambda-\omega)^n\|R(\lambda,t_1,\ldots,t_n)\|_{B(Y)} \le \|P_2(t_1)\|_{B(Y)}\|P_2^{-1}(t_n)\|_{B(Y)} \le K^2.$$

(iii). Clear. □

**Lemma 2.5.** Let $P_i(\cdot)$ satisfy (C1 – C3) on $(Y_i^0, \|\cdot\|_{Y_i^0})$ and $(Y_i, \|\cdot\|_{Y_i})$, $i = 1, 2$. Then

$$\begin{bmatrix} P_1(\cdot) & 0 \\ 0 & P_2(\cdot) \end{bmatrix}$$

satisfy (C1 – C3) on $Y_0 = (Y_1^0 \times Y_2^0, \|\cdot\|_{Y_0})$ and $Y = (Y_1 \times Y_2, \|\cdot\|_Y)$, with $\|\cdot\|_Y = \max\{\|\cdot\|_{Y_1}, \|\cdot\|_{Y_2}\}$ and $\|\cdot\|_{Y_0} = \max\{\|\cdot\|_{Y_1^0}, \|\cdot\|_{Y_2^0}\}$.

**Proof.** By a routine check. □

In the following, the norm on a product space is always taken as the maximum of individual norms. Now we can prove the following results concerning the leading operators in Eq. (1.5) and Eq. (1.7).

**Theorem 2.6.** Let $A(\cdot)$ satisfy (C4) on $E_0$ and $E$. Let $K(t), G(t,t) \in B(E)$ with $\|K(t)\|_{B(E)} + \|G(t,t)\|_{B(E)} \le \beta < 1$, $\beta > 0$ is a constant, and $K(\cdot), G(\cdot,\cdot) \in C^1([0,T], B(E))$ such that for each $k \in N$ there exists $K_k(\cdot), G_k(\cdot) \in C^4([0,T], B(E))$ with

$$\lim_{k\to\infty}\|K(\cdot) - K_k(\cdot)\|_{C^1([0,T],B(E))} = 0, \quad \lim_{k\to\infty}\|G(\cdot,\cdot) - G_k(\cdot)\|_{C^1([0,T],B(E))} = 0.$$

Then

$$Q(t) \equiv \begin{bmatrix} K(t) & A(t) \\ K(t) + G(t,t) & A(t) \end{bmatrix} \tag{2.4}$$

satisfies (C4) on $Y_0 = E \times E_0$ and $Y = E \times E$.

**Proof.** We have

$$Q(t) = \begin{bmatrix} 0 & A(t) \\ 0 & A(t) \end{bmatrix} + \begin{bmatrix} K(t) & 0 \\ K(t) + G(t,t) & 0 \end{bmatrix} \equiv P_0(t) + P_1(t).$$

Next, let $I$ be the identity operator on $E$, then

$$\begin{aligned}
P_0(t) \;&=\; \begin{bmatrix} 0 & A(t) \\ 0 & A(t) \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & A(t) \end{bmatrix}\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \\
&\equiv\; P_2 P(t) P_2^{-1}.
\end{aligned}$$

Note that by Lemma 2.5, $P(\cdot)$ satisfies (C1 – C3) on $Y_0 = E \times E_0, Y = E \times E$. Also note that $P_2, P_2^{-1} \in B(Y_0) \cap B(Y)$ are independent of $t$, so Lemma 2.4 (iii) implies that $P_0(\cdot)$ satisfies (C1 – C3) on $Y_0, Y$. Next, observe that $P_1(\cdot)$ satisfies conditions in Lemma 2.4 (i), so $Q(\cdot)$ satisfies (C1 – C3) on $Y_0, Y$ by using Lemma 2.4 again. Finally, it can be checked that $Q(\cdot)$ also satisfies (C4). $\square$

Now we define according to Eq.(1.5) and Eq.(1.6),

$$H(t,s) \equiv \begin{bmatrix} 0 & 0 \\ G_t(t,s) & 0 \end{bmatrix}, \quad g(t) \equiv \begin{bmatrix} f(t) \\ f(t) + \int_{-\infty}^{0} G_t(t,s)\varphi(s)ds \end{bmatrix}, \quad t \geq 0, \ s \leq t. \tag{2.5}$$

Let $X_0, X$ be Banach spaces. In the following, we use $F_X$ to denote a subspace of $BU(X)$ (the space of all bounded uniformly continuous functions on $[0, \infty)$ into Banach space $X$). It is assumed that $F_X$ is a Banach space with a norm stronger than the sup norm on $BU(X)$, (see [5]). Further, assume that for each $t \geq 0, H(t+\cdot, t)x \in F_X$ for every $x \in X_0 \subset X$ where $H(t+\cdot, t)x(s) = H(t+s, t)x$ for $s \geq 0$. This then defines an operator $H(t) : X_0 \to F_X$. Furthermore, $D_s$ denotes the generator of translation semigroup $(T(t)f)(\cdot) = f(t+\cdot)$ on $F_X$ with domain $D(D_s)$, $D(D_s)$ equipped with the graph norm is a Banach space. $\delta f = f(0)$ for $f \in F_X$. Now, we have

**Theorem 2.7.** Assume that $Q(\cdot)$ satisfies (C4) on $X_0$ and $X$. Assume further that $\|\cdot\|_{F_X} = (1+\gamma)\|\cdot\|_{BU(X)}, \gamma > 0$ is a constant, (so $\|\cdot\|_{F_X}$ is stronger than $\|\cdot\|_{BU(X)}$, [5]), and that for $t \geq 0, \|H(t)x\|_{F_X} \leq (1+\gamma)^{-1}\|x\|_X, x \in X_0$. Finally, assume that $H(\cdot) \in C^1([0,T], B(X_0, F_X))$ and there exist $H_k(\cdot) \in C^4([0,T], B(X_0, F_X))$ such that $\lim_{k\to\infty} \|H - H_k\|_{C^1([0,T],B(X_0,F_X))} = 0$. Then

$$P(t) \equiv \begin{bmatrix} Q(t) & \delta \\ H(t) & D_s \end{bmatrix}$$

satisfies (C4) on $Y_0 = X_0 \times D(D_s)$ and $Y = X \times F_X$.

**Proof.** Since $D_s$ generates a strongly continuous semigroup on $F_X$, $D_s$ satisfies (C1 – C3) on $D_s, F_X$. Thus Lemma 2.5 implies that

$$P(t) \equiv \begin{bmatrix} Q(t) & 0 \\ 0 & D_s \end{bmatrix}$$

satisfies (C1 – C3) on $Y_0 = X_0 \times D(D_s), Y = X \times F_X$. Next, we have for $(x, f) \in Y$,

$$\begin{aligned} &\left\| \begin{bmatrix} 0 & \delta \\ H(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} \right\|_Y = \left\| \begin{bmatrix} f(0) \\ H(t)x \end{bmatrix} \right\|_Y \\ &= \max\left\{\|f(0)\|_X, \|H(t)x\|_{F_X}\right\} \leq \max\left\{\|f(\cdot)\|_{BU(X)}, (1+\gamma)^{-1}\|x\|_X\right\} \\ &= \max\left\{(1+\gamma)^{-1}\|f(\cdot)\|_{F_X}, (1+\gamma)^{-1}\|x\|_X\right\} = (1+\gamma)^{-1}\left\| \begin{bmatrix} x \\ f \end{bmatrix} \right\|_Y. \end{aligned}$$

Thus

$$\left\| \begin{bmatrix} 0 & \delta \\ H(t) & 0 \end{bmatrix} \right\|_{B(Y)} \leq (1+\gamma)^{-1} < 1.$$

Now we can apply Lemmas 2.4 (i) to show that $P(\cdot)$ satisfies (C1 – C3) on $Y_0, Y$. It is also clear that (C4) is true for $P(\cdot)$. $\square$

Now, we can study the strict solutions of Eq.(1.4) on $E_0, E$. (See Definition 2.2.) Note that $BU(E), F_E$ will be defined accordingly. (See the notations before Theorem 2.7.) Assume that for each $t \geq 0, G_t(t+\cdot, t)e \in F_E$ for every $e \in E$ where $G_t(t+\cdot, t)e(s) = G_t(t+s, t)e$ for $s \geq 0$. This then defines an operator $G_t(t) : E \to F_E$. We now make the following assumptions.

(A1). $A(\cdot), K(\cdot), G(\cdot,\cdot)$ satisfy the conditions in Theorem 2.6 on $E_0, E$. Assume that $\|\cdot\|_{F_E} = (1+\gamma)\|\cdot\|_{BU(E)}, \gamma > 0$ is a constant, and that for $t \geq 0, \|G_t(t)e\|_{F_E} \leq (1+\gamma)^{-1}\|e\|_E, e \in E$. Finally, assume that $G_t(\cdot) \in C^1([0,T], B(E, F_E))$ and there exist $G_t^k(\cdot) \in C^4([0,T], B(E, F_E))$ such that $\lim_{k\to\infty} \|G_t - G_t^k\|_{C^1([0,T],B(E,F_E))} = 0$.

(A2). $f \in W^{1,p}([0,T], E)$. $(1 \leq p < \infty.)$ $\int_{-\infty}^0 G_t(t,s)\varphi(s)ds \in W^{1,p}([0,T], E)$ if $\varphi(\cdot) \in L^1((-\infty, 0], E)$.

(A3). $\varphi(\cdot) \in L^1((-\infty, 0], E)$. $w_0 \equiv \varphi(0) + \int_{-\infty}^0 G(0,s)\varphi(s)ds \in E_0$, and $\Big[K(0) + G(0,0)\Big]\varphi(0) + A(0)w_0 + f(0) + \int_{-\infty}^0 G_t(0,s)\varphi(s)ds \in \overline{E_0}$.

We will prove that Eq.(1.4) has a strict solution under these assumptions.

**Theorem 2.8.** Let Assumptions (A1 – A2) be satisfied. Then given $\big(\varphi(0), \varphi(\cdot)\big)$ satisfying (A3), Eq.(1.4) has a strict solution on $E_0$ and $E$.

**Proof.** Let $X_0 = E \times E_0, X = E \times E, F_X = F_E \times F_E$, and let $Q(t), H(t,s), g(t)$ be given by (2.4) and (2.5). By Assumption (A1) and Theorem 2.6, $Q(\cdot)$ satisfies (C4) on $X_0, X$. Next, define

$$y(t) \equiv \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, h(t) \equiv \begin{bmatrix} g(t) \\ 0 \end{bmatrix}, P(t) \equiv \begin{bmatrix} Q(t) & \delta \\ H(t) & D_s \end{bmatrix}, \quad t \in [0,T],$$

on $Y_0 = X_0 \times D(D_s), Y = X \times F_X$ according to Eq.(1.7) and Eq.(2.1). Then for $(f_1, f_2) \in F_X$,

$$\begin{aligned} \left\|\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right\|_{F_X} &= \max\Big\{\|f_1\|_{F_E}, \|f_2\|_{F_E}\Big\} \\ &= \max\Big\{(1+\gamma)\|f_1\|_{BU(E)}, (1+\gamma)\|f_2\|_{BU(E)}\Big\} \\ &= (1+\gamma)\left\|\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right\|_{BU(X)}, \end{aligned}$$

and for $x = (e, e_0) \in X_0 = E \times E_0$,

$$\begin{aligned} \|H(t)x\|_{F_X} &= \left\|\begin{bmatrix} 0 & 0 \\ G_t(t) & 0 \end{bmatrix}\begin{bmatrix} e \\ e_0 \end{bmatrix}\right\|_{F_X} = \|G_t(t)e\|_{F_E} \\ &\leq (1+\gamma)^{-1}\|e\|_E \leq (1+\gamma)^{-1}\|x\|_X. \end{aligned}$$

Other conditions in Theorem 2.7 can also be checked. Hence $P(\cdot)$ satisfies (C4) on $X_0 \times D(D_s), X \times F_X$. Finally, from Eq.(1.5) and Eq.(1.7), one has

$$y_0 \equiv \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad x_0 \equiv \begin{bmatrix} \varphi(0) \\ \varphi(0) + \int_{-\infty}^0 G(0,s)\varphi(s)ds \end{bmatrix}.$$

So with assumptions (A2 – A3), one can check that

$$h \in W^{1,p}([0,T], X \times F_X), \quad y_0 \in X_0 \times D(D_s), \quad P(0)y_0 + h(0) \in \overline{X_0 \times D(D_s)} = \overline{X_0} \times F_X.$$

Thus Theorem 2.3 implies that Eq. (1.7) (and hence Eq. (1.5)) has a strict solution. Therefore the first component of Eq. (1.5) (which comes from the first component of Eq. (1.7)) gives rise to a strict solution of Eq. (1.4). □

# 3 AN APPLICATION IN VISCOELASTICITY.

In this section, we will put some conditions on $\rho(t,x), E(t,x), k(t,x), b(t,s), h(t,x), \phi(t,x)$ so as to verify that assumptions in (A1 – A3) in Section 2 are satisfied for Eq. (1.3).

We let $C[0,1]$ be the space of all continuous functions on $[0,1]$ with the sup norm. Let $E = C[0,1]\times C[0,1]$ with $\|(u,v)\|_E = \max\{\|u\|_{C[0,1]}, \|v\|_{C[0,1]}\}$. If $f \in C[0,1]$, then $f$ can be regarded as an bounded linear operator on $C[0,1]$ by defining $(f(g))(x) = f(x)g(x), g \in C[0,1], x \in [0,1]$. Define

$$Y_0 \equiv \Big\{(y,z) \in E : y,z \in C^1[0,1], y(0)+z(0) = y(1)+z(1) = 0\Big\}. \tag{3.1}$$

Then it has been shown in Grimmer and Sinestrari [9] that

$$P_0 \equiv \begin{bmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{bmatrix} \tag{3.2}$$

with domain $Y_0$ satisfies $\lambda \in \rho(P_0)$ and $\|\lambda(\lambda - P_0)^{-1}\|_{B(E)} \le 1$ if $\lambda > 0$. Next, define

$$E_0 \equiv \Big\{(v,w) \in E : v,w \in C^1[0,1], w(0) = w(1) = 0\Big\}, \tag{3.3}$$

and define the norms on $Y_0, E_0$ by

$$\|(y,z)\|_{Y_0} = \|(y,z)\|_E + \|P_0(y,z)^T\|_E, \tag{3.4}$$

$$\|(v,w)\|_{E_0} = \|(v,w)\|_E + \|\begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix}(v,w)^T\|_E, \tag{3.5}$$

where $T$ indicates transpose. Then by a suitable choice of a $2\times 2$ matrix, we find that if $\rho(t,\cdot), E(t,\cdot) \in C^1[0,1], t \in [0,T]$, then

$$P_2(t) \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ \sqrt{\frac{E(t,\cdot)}{\rho(t,\cdot)}} & \sqrt{\frac{E(t,\cdot)}{\rho(t,\cdot)}} \end{bmatrix} : Y_0 \to E_0, \tag{3.6}$$

$$P_2^{-1}(t) \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \sqrt{\frac{\rho(t,\cdot)}{E(t,\cdot)}} \\ -1 & \sqrt{\frac{\rho(t,\cdot)}{E(t,\cdot)}} \end{bmatrix} : E_0 \to Y_0, \tag{3.7}$$

and the leading operator in Eq.(1.3) can be written as

$$\begin{aligned} A(t) &\equiv \begin{bmatrix} 0 & \partial_x \\ \frac{E(t,\cdot)}{\rho(t,\cdot)}\partial_x + \frac{E_x(t,\cdot)}{\rho(t,\cdot)} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \partial_x \\ \frac{E(t,\cdot)}{\rho(t,\cdot)}\partial_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{E_x(t,\cdot)}{\rho(t,\cdot)} & 0 \end{bmatrix} \\ &= P_2(t)\Big[\sqrt{\frac{E(t,\cdot)}{\rho(t,\cdot)}}P_0\Big]P_2^{-1}(t) + \begin{bmatrix} 0 & -\sqrt{\frac{E(t,\cdot)}{\rho(t,\cdot)}}\Big(\frac{\partial}{\partial x}\sqrt{\frac{\rho(t,\cdot)}{E(t,\cdot)}}\Big) \\ \frac{E_x(t,\cdot)}{\rho(t,\cdot)} & 0 \end{bmatrix} \\ &\equiv \widetilde{P_0}(t) + \widetilde{P_1}(t). \end{aligned} \tag{3.8}$$

So we want to prove that $P(t) \equiv \sqrt{\frac{E(t,\cdot)}{\rho(t,\cdot)}} P_0$ satisfies (C1 – C3) and that other perturbation conditions in Lemma 2.4 are satisfied so as to show that $A(\cdot)$ satisfies (C1 – C3). We need the following lemma, its proof is a modification of the one in Grimmer and Sinestrari [9].

**Lemma 3.1.** Let $q(\cdot) \in C[0,1]$ with $q(x) > 0, x \in [0,1]$. Then for $\lambda > 0, \lambda \in \rho(q(\cdot)P_0)$ and

$$\|\lambda(\lambda - q(\cdot)P_0)^{-1}\|_{B(E)} \leq 1. \tag{3.9}$$

**Proof.** Let $\lambda > 0$. $U = (y, z) \in D(P_0)$ is a solution of $\lambda U^T - q(\cdot)P_0U^T = \lambda e, e = (f, g) \in E$, if and only if $y, z \in C^1[0,1]$ and

$$\begin{cases} \lambda y(x) - q(x)y'(x) = \lambda f(x), \\ \lambda z(x) + q(x)z'(x) = \lambda g(x), \\ y(0) + z(0) = y(1) + z(1) = 0. \end{cases} \tag{3.10}$$

Define $k(x) = \lambda/q(x)(> 0), x \in [0,1]$. Then Eq.(3.10) has a unique solution given by

$$y(x) = e^{\int_0^x k(s)ds} y(0) - \int_0^x e^{\int_s^x k(h)dh} k(s)f(s)ds, \tag{3.11}$$

$$z(x) = e^{-\int_0^x k(s)ds} z(0) + \int_0^x e^{-\int_s^x k(h)dh} k(s)g(s)ds. \tag{3.12}$$

Consider $y(x) + z(x)$ and use boundary conditions in (3.10), we obtain

$$y(0)=\Big(e^{\int_0^1 k(s)ds} - e^{-\int_0^1 k(s)ds}\Big)^{-1} \int_0^1 \Big[e^{\int_s^1 k(h)dh} k(s)f(s) - e^{-\int_s^1 k(h)dh} k(s)g(s)\Big] ds. \tag{3.13}$$

Thus

$$\begin{aligned} |y(0)| &\leq \Big(e^{\int_0^1 k(s)ds} - e^{-\int_0^1 k(s)ds}\Big)^{-1} \int_0^1 \Big[e^{\int_s^1 k(h)dh} k(s) + e^{-\int_s^1 k(h)dh} k(s)\Big] ds \|e\|_E \\ &= \|e\|_E. \end{aligned} \tag{3.14}$$

Next, from (3.11) and (3.13), we obtain

$$\begin{aligned} y(1) &= e^{\int_0^1 k(s)ds} y(0) - \int_0^1 e^{\int_s^1 k(h)dh} k(s)f(s)ds \\ &= \Big(e^{\int_0^1 k(s)ds} - e^{-\int_0^1 k(s)ds}\Big)^{-1} \Big\{e^{-\int_0^1 k(s)ds} \int_0^1 e^{\int_s^1 k(h)dh} k(s)f(s)ds \\ &\quad - e^{\int_0^1 k(s)ds} \int_0^1 e^{-\int_s^1 k(h)dh} k(s)g(s)ds\Big\}. \end{aligned} \tag{3.15}$$

So similar to (3.14), one has $|y(1)| \leq \|e\|_E$. Now let $x_0 \in [0,1]$ be such that $|y(x_0)| = \|y\|_{C[0,1]}$. If $0 < x_0 < 1$, then $y'(x_0) = 0$. So from (3.10),

$$\|y\|_{C[0,1]} = |y(x_0)| = |f(x_0)| \leq \|e\|_E.$$

On the other hand, if $x_0 = 0$ or $x_0 = 1$, then (3.14) and (3.15) imply $\|y\|_{C[0,1]} \leq \|e\|_E$.

Similarly, from the boundary conditions in (3.10), one has $\|z\|_{C[0,1]} \leq \|e\|_E$. Therefore $\|\lambda(\lambda - q(\cdot)P_0)^{-1}e\|_E = \|U\|_E \leq \|e\|_E$ and hence the proof is completed. □

Next, we show that (A1 – A3) in Section 2 are satisfied for Eq.(1.3) with the following assumptions.

$(\widetilde{A1})$. $\rho(t,\cdot), E(t,\cdot) \in C^1[0,1], t \in [0,T]$ with $\rho(t,x), E(t,x) > 0$. There is a constant $c > 0$ with $c^{-1} \le \rho(t,x), E(t,x), \rho_x(t,x) \le c, (t,x) \in [0,T] \times [0,1]$. For any $x \in [0,1]$, $\frac{E(t,x)}{\rho(t,x)}$ is nondecreasing in $t \in [0,T]$. There is a constant $0 < \beta < 1$ such that $\max_{(t,x)\in[0,T]\times[0,1]} \left\{ \frac{E_x(t,x)}{\rho(t,x)}, \sqrt{\frac{E(t,x)}{\rho(t,x)}} \left( \frac{\partial}{\partial x} \sqrt{\frac{\rho(t,x)}{E(t,x)}} \right), \frac{|b(t,t)|}{E(t,x)}, \frac{|k(t,x)|}{\rho(t,x)} \right\} \le \beta < 1$. $\frac{\partial}{\partial t}\frac{k(t,x)}{\rho(t,x)}$ and $\frac{\partial}{\partial t}\frac{b(t,t)}{E(t,x)}$ are continuous in $(t,x) \in [0,T] \times [0,1]$. There is a constant $\gamma > 0$ such that for $t \ge 0, \max_{(s,x)\in[0,\infty)\times[0,1]} |\frac{\partial}{\partial t}\frac{b(t+s,t)}{E(t,x)}| \le (1+\gamma)^{-2}$ and $\frac{\partial^2}{\partial t^2}\frac{b(t+s,t)}{E(t,x)}$ is bounded and uniformly continuous for $(t,s,x) \in [0,T] \times [0,\infty) \times [0,1]$. $\|\cdot\|_{F_{C[0,1]}} = (1+\gamma)\|\cdot\|_{BU(C[0,1])}$.

$(\widetilde{A2})$. $\frac{h(t,\cdot)}{\rho(t,\cdot)} \in W^{1,p}([0,T], C[0,1]), 1 \le p < \infty$. $\frac{\partial^2}{\partial t^2}\frac{b(t,s)}{E(t,x)}$ is bounded and uniformly continuous for $(t,s,x) \in [0,T] \times (-\infty,0] \times [0,1]$.

$(\widetilde{A3})$. Define $C_0[0,1] \equiv \{f \in C[0,1] : f(0) = f(1) = 0\}$. Then

$$\begin{aligned}\phi_t, \phi_x \in L^1((-\infty,0], C[0,1]), \phi(0,0) = \phi(0,1) = 0,\\ \phi_t(0,\cdot) \in C^1[0,1] \cap C_0[0,1],\\ \phi_x(0,\cdot) + \int_{-\infty}^0 \frac{b(0,s)}{E(0,\cdot)}\phi_x(s,\cdot)ds \in C^1[0,1] \cap C_0[0,1],\\ \int_{-\infty}^0 \phi_x(s,\cdot)\frac{\partial}{\partial t}\frac{b(0,s)}{E(0,\cdot)}ds \in C[0,1],\\ E(0,\cdot)\frac{\partial}{\partial x}\Big[\phi_x(0,\cdot) + \int_{-\infty}^0 \frac{b(0,s)}{E(0,\cdot)}\phi_x(s,\cdot)ds\Big] + h(0,\cdot) \in C_0[0,1].\end{aligned}$$

**Lemma 3.2.** Let $(\widetilde{A1})$ be satisfied. Then $A(\cdot)$ defined by (3.8) satisfies (C4) in Section 2 on $E_0$ and $E$.

**Proof.** First, note that with $(\widetilde{A1})$ and Lemma 3.1, $P(t) \equiv \sqrt{\frac{E(t,\cdot)}{\rho(t,\cdot)}}P_0$ in (3.8) satisfies $(\mathrm{C1}-\mathrm{C3})$ with $M = 1$. Next, consider $P_2(\cdot), P_2^{-1}(\cdot)$ defined in (3.6) and (3.7). For $t_2 \le t_1$,

$$\begin{aligned} P_2^{-1}(t_1)P_2(t_2) &= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{\frac{\rho(t_1,\cdot)}{E(t_1,\cdot)}} \\ -1 & \sqrt{\frac{\rho(t_1,\cdot)}{E(t_1,\cdot)}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \sqrt{\frac{E(t_2,\cdot)}{\rho(t_2,\cdot)}} & \sqrt{\frac{E(t_2,\cdot)}{\rho(t_2,\cdot)}} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 + \sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}} & \sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}} - 1 \\ \sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}} - 1 & 1 + \sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}} \end{bmatrix}. \end{aligned}$$

From $(\widetilde{A1})$, $\sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}} \le 1$, then

$$\|P_2^{-1}(t_1)P_2(t_2)\|_{B(E)} \le \frac{1}{2}\max_{x\in[0,1]}\Big\{1 + \sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}} + 1 - \sqrt{\frac{\rho(t_1,\cdot)E(t_2,\cdot)}{E(t_1,\cdot)\rho(t_2,\cdot)}}\Big\} = 1.$$

It can also be checked that $P_2(t) \in B(Y_0,E_0), P_2^{-1}(t) \in B(E_0,Y_0)$ are bounded in $t \in [0,T]$. Therefore conditions in Lemma 2.4 (ii) are satisfied and hence $\widetilde{P_0}(t)$ defined in (3.8) satisfies $(\mathrm{C1}-\mathrm{C3})$ on $E_0, E$. (Use $Y = E, Z_0 = E_0$ in Lemma 2.4 (ii).)

Next, observe that $\widetilde{P_1}(t)$ defined in (3.8) satisfies conditions in Lemma 2.4 (i), so that $A(\cdot)$ satisfies (C1 – C3). Finally it is clear that (C4) is also true for $A(\cdot)$. $\square$

Now we check other conditions. We define according to Eq.(1.3) and Eq.(1.4),

$$G(t,s) \equiv \begin{bmatrix} \frac{b(t,s)}{E(t,\cdot)} & 0 \\ 0 & 0 \end{bmatrix}, \quad K(t) \equiv \begin{bmatrix} 0 & 0 \\ 0 & \frac{-k(t,\cdot)}{\rho(t,\cdot)} \end{bmatrix}, \quad t \in [0,T], \tag{3.16}$$

$$f(t) \equiv \begin{bmatrix} 0 \\ \frac{h(t,\cdot)}{\rho(t.\cdot)} \end{bmatrix}, \quad t \in [0,T], \quad \varphi(s) \equiv \begin{bmatrix} \phi_x(s) \\ \phi_t(s) \end{bmatrix}, \quad s \leq 0. \tag{3.17}$$

With the above definitions, we have

**Lemma 3.3.** Let $(\widetilde{A1})$ be satisfied. Then $A(t), K(t), G(t,s)$ defined by (3.8) and (3.16) satisfies (A1) in Section 2 on $E_0$ and $E$.

**Proof.** For $e = (e_1, e_2) \in E = C[0,1] \times C[0,1]$, one has

$$\begin{aligned} \|G_t(t)e\|_{F_E} &= \| \begin{bmatrix} \frac{\partial}{\partial t}\frac{b(t+s,t)}{E(t,\cdot)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \|_{F_E} \\ &= \| \frac{\partial}{\partial t}\frac{b(t+s,t)}{E(t,\cdot)} e_1 \|_{F_{C[0,1]}} = (1+\gamma) \| \frac{\partial}{\partial t}\frac{b(t+s,t)}{E(t,\cdot)} e_1 \|_{BU(C[0,1])} \\ &\leq (1+\gamma)^{-1} \|e_1\|_{C[0,1]} \leq (1+\gamma)^{-1} \|e\|_E. \end{aligned}$$

Other conditions can be checked accordingly. $\square$

Finally, observe that Assumptions $(\widetilde{A2}, \widetilde{A3})$ are made according to Assumptions (A2, A3). So by a routine check (details are omitted for simplicity), one shows that if Assumptions $(\widetilde{A2}, \widetilde{A3})$ are true, then Assumptions (A2, A3) are true. Thus we obtain the following result concerning the strict solutions of Eq. (1.3), and hence solutions of Eq. (1.2).

**Theorem 3.4.** Let Assumptions $(\widetilde{A1}, \widetilde{A2})$ be satisfied. Then given $\big(\phi(0,\cdot), \phi, \phi_x(0,\cdot), \phi_t(0,\cdot), \phi_x, \phi_t\big)$ satisfying $(\widetilde{A3})$, Eq. (1.3) (when treated as a form of Eq. (1.4)) has a strict solution. Hence there is a function $u(t,x)$ satisfies Eq. (1.2) and such that

$$u_{tt}, \; u_{xt}, \; \frac{\partial}{\partial x}\Big[u_x(t,x) + \frac{1}{E(t,x)} \int_{-\infty}^{t} b(t,s) u_x(s,x) ds\Big] \in C([0,T] \times [0,1], \Re).$$

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