SINGULAR PERTURBATIONS OF INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract

Let $\varepsilon > 0$ and consider

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) &= Au(t;\varepsilon) + \int_0^t K(t-s)Au(s;\varepsilon)ds + f(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) &= u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned}$$

and

$$w'(t) = Aw(t) + \int_0^t K(t-s)Aw(s)ds + f(t), \;\; t \geq 0, \;\; w(0) = w_0,$$

in a Banach space X when $\varepsilon \to 0$. Here A is the generator of a strongly continuous cosine family and a strongly continuous semigroup, and $K(t)$ is a bounded linear operator for $t \geq 0$. With some convergence conditions on initial data and $f(t;\varepsilon)$ and smoothness conditions on $K(\cdot)$, we prove that if $\varepsilon \to 0$, then $u(t;\varepsilon) \to w(t)$ in X uniformly for $t \in [0,T]$ for any fixed $T > 0$. We will apply this to an equation in viscoelasticity.

1 INTRODUCTION.

We study integrodifferential equations

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) &= Au(t;\varepsilon) + \int_0^t K(t-s)Au(s;\varepsilon)ds + f(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) &= u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned} \tag{1.1}$$

and

$$w'(t) = Aw(t) + \int_0^t K(t-s)Aw(s)ds + f(t), \;\; t \geq 0, \;\; w(0) = w_0, \tag{1.2}$$

in a Banach space X, with A the generator of a strongly continuous cosine family and a strongly continuous semigroup, and $K(t)$ a bounded linear operator for $t \geq 0$. We regard

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Eq.(1.2) as the limiting equation of Eq.(1.1) as $\varepsilon \to 0$. Now, Eq.(1.2) is of lower order of derivative (in t), in this sense we say that we are dealing with the singular perturbation problems.

There are many studies on singular perturbations, see e.g., Goldstein [6], Hale and Raugel [10], Smith [13], Grimmer and Liu [8], and the references therein. Since this work was influenced by Fattorini [5], we only state some results of [5].

Fattorini [F] considered the singular perturbations for

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) &= Au(t;\varepsilon) + f(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) &= u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned} \tag{1.3}$$

and

$$w'(t) = Aw(t) + f(t), \;\; t \geq 0, \;\; w(0) = w_0, \tag{1.4}$$

with A the generator of a strongly continuous cosine family and a strongly continuous semigroup in a Banach space X and proved that:

> For any $T > 0$, if $f(\cdot;\varepsilon) \to f$ in $L^1([0,T],X)$ and $u_0(\varepsilon) \to w_0$, $\varepsilon^2 u_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, then $u(t;\varepsilon) \to w(t)$ in X uniformly for $t \in [0,T]$ as $\varepsilon \to 0$.

We will prove here with some smoothness conditions on $K(\cdot)$ that exactly the same statements as above hold for Eqs.(1.1) and (1.2). The methods we will use in studying the singular perturbations for integrodifferential equations are as follows: We first use the technique introduced in [1, 2, 11, 12] to change Eqs.(1.1) and (1.2) into equations that look like Eqs.(1.3) and (1.4), and then estimate $u(t;\varepsilon) - w(t)$. Note that $u(\cdot;\varepsilon) - w(\cdot)$ will also appear as an integrand, so Gronwall's inequality is used to solve the problem. Finally we apply this result to an equation in viscoelasticity.

2 SINGULAR PERTURBATIONS.

In this paper we make the following hypotheses:

(H1). Operator A generates a strongly continuous cosine family $C(\cdot)$ and a strongly continuous semigroup $S(\cdot)$. (See [5].)

(H2). For $t \geq 0, K(t), K'(t), K''(t) \in B(X), (B(X)$ = space of all bounded linear operators on $X)$. For $x \in X, Kx, K'x, K''x \in L^1_{loc}(R^+, X)$. Here K', K'' are the strong derivatives.

(H3). $f(\cdot;\varepsilon), f \in C^1(R^+, X)$, where $\varepsilon > 0, R^+ = [0, \infty)$.

We say that $u : R^+ \to X$ is a solution of Eq.(1.1) if $u \in C^2(R^+, X), u(t) \in D(A)$ (domain of A) for $t \geq 0$ and Eq.(1.1) is satisfied on R^+. Solutions of Eq.(1.2) are defined in a similar way. In order to verify the existence of solutions of Eq.(1.1) we change it to another more common form. (See [5].) Let

$$u(t;\varepsilon) = e^{-t/2\varepsilon^2} v(t/\varepsilon).$$

Then Eq.(1.1) can be replaced by

$$v''(t/\varepsilon) = \Big(A + \frac{1}{4\varepsilon^2}\Big) v(t/\varepsilon) + \int_0^t K(t-s) e^{(t-s)/2\varepsilon^2} A v(s/\varepsilon) ds + e^{t/2\varepsilon^2} f(t;\varepsilon).$$

Now let $h = t/\varepsilon$ and then change h to t to get

$$\begin{aligned} v''(t) &= \Big(A + \frac{1}{4\varepsilon^2}\Big) v(t) + \int_0^t \hat{K}(t-s) A v(s) ds + \hat{f}(t), \qquad (2.1) \\ v(0;\varepsilon) &= u_0(\varepsilon), \;\; v'(0;\varepsilon) = \frac{1}{2\varepsilon} u_0(\varepsilon) + \varepsilon u_1(\varepsilon), \end{aligned}$$

where $\Big(A + \frac{1}{4\varepsilon^2}\Big)$ generates a strongly continuous cosine family and

$$\hat{K}(t) = \varepsilon K(\varepsilon t) e^{t/2\varepsilon}, \quad \hat{f}(t) = f(\varepsilon t;\varepsilon) e^{t/2\varepsilon}, \;\; t \geq 0.$$

Note that the existence and uniqueness of solutions of Eqs.(2.1) and (1.2) were obtained in [3, 4, 7, 14, 15], and we are only interested in singular perturbations in this paper, so we may assume that Eqs.(1.1) and (1.2) have unique solutions $u(t;\varepsilon)$ and $w(t)$ respectively for every $\varepsilon > 0$.

Now we can state and prove the following result concerning the convergence of solutions, with the following hypotheses:

(H4). $u_0(\varepsilon), w_0 \in D(A), u_0(\varepsilon) \to w_0, \varepsilon^2 u_1(\varepsilon) \to 0,$ as $\varepsilon \to 0$.

(H5). For any $T > 0, f(\cdot;\varepsilon) \to f(\cdot)$ in $L^1([0,T], X)$ as $\varepsilon \to 0$.

Theorem 2.1. Assume that hypotheses (H1) – (H5) are satisfied. Then for any $T > 0, u(t;\varepsilon) \to w(t)$ in X uniformly for $t \in [0,T]$, as $\varepsilon \to 0$.

Proof. Define

$$R * H(t) = \int_0^t R(t-s) H(s) ds \;\text{ and }\; \delta * H = H.$$

Then we can find the solution F of $F + K + F * K = 0$. (See [1, 2, 11, 12].) So that

$$(\delta + F) * (\delta + K) = \delta. \qquad (2.2)$$

Now write (1.1) as

$$\varepsilon^2 u''(\varepsilon) + u'(\varepsilon) = (\delta + K) * A u(\varepsilon) + f(\varepsilon).$$

Then we have

$$(\delta + F) * \Big[\varepsilon^2 u''(\varepsilon) + u'(\varepsilon)\Big] = Au(\varepsilon) + (\delta + F) * f(\varepsilon).$$

Hence

$$\varepsilon^2 u''(\varepsilon) + u'(\varepsilon) = Au(\varepsilon) + (\delta + F) * f(\varepsilon) - F * \Big[\varepsilon^2 u''(\varepsilon) + u'(\varepsilon)\Big].$$

Integration by parts yields

$$F * u'(t;\varepsilon) = \int_0^t F'(t-s)u(s;\varepsilon)ds + F(0)u(t;\varepsilon) - F(t)u_0(\varepsilon),$$

$$F * u''(t;\varepsilon) = \int_0^t F''(t-s)u(s;\varepsilon)ds + F(0)u'(t;\varepsilon) - F(t)u_1(\varepsilon) + F'(0)u(t;\varepsilon) - F'(t)u_0(\varepsilon).$$

Therefore Eq.(1.1) can be replaced by

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) &= Au(t;\varepsilon) + \hat{f}(t;\varepsilon), \quad t \ge 0, \\ u(0;\varepsilon) &= u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} \hat{f}(t;\varepsilon) &= (\delta + F) * f(t;\varepsilon) - F * \Big[\varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon)\Big] \qquad (2.4) \\ &= (\delta + F) * f(t;\varepsilon) - \int_0^t F'(t-s)u(s;\varepsilon)ds - F(0)u(t;\varepsilon) + F(t)u_0(\varepsilon) - \\ &\quad \varepsilon^2\Big[\int_0^t F''(t-s)u(s;\varepsilon)ds + F'(0)u(t;\varepsilon) - F'(t)u_0(\varepsilon) + F(0)u'(t;\varepsilon) - F(t)u_1(\varepsilon)\Big]. \end{aligned}$$

Similarly, Eq.(1.2) can be replaced by

$$w'(t) = Aw(t) + \hat{f}(t), \;\; t \ge 0, \;\; w(0) = w_0, \tag{2.5}$$

with

$$\begin{aligned} \hat{f}(t) &= (\delta + F) * f(t) - F * w'(t) \\ &= (\delta + F) * f(t) - \int_0^t F'(t-s)w(s)ds - F(0)w(t) + F(t)w_0. \qquad (2.6) \end{aligned}$$

So we can use results in [5] to get for $t \ge 0$,

$$\begin{aligned} w(t) &= S(t)w_0 + \int_0^t S(t-s)\hat{f}(s)ds, \\ u(t;\varepsilon) &= e^{-t/2\varepsilon^2}C(t/\varepsilon)u_0(\varepsilon) + \frac{1}{2}R(t,\varepsilon)u_0(\varepsilon) \\ &\quad + G(t,\varepsilon)\Big[\frac{1}{2}u_0(\varepsilon) + \varepsilon^2 u_1(\varepsilon)\Big] + \int_0^t G(t-s)\hat{f}(s,\varepsilon)ds, \end{aligned}$$

where $S(\cdot), C(\cdot)$ are given in (H1), $R(\cdot;\varepsilon), G(\cdot;\varepsilon)$ are linear operators defined in [5] using the Bessel functions, and they have the following properties: For some constants $\alpha, \omega > 0$, (P1). $\|C(t)\|, \|S(t)\| \le \alpha e^{\omega^2 t}, t \ge 0, \varepsilon > 0.$

(P2). $\|G(t;\varepsilon)\|, \varepsilon^2\|G'(t;\varepsilon)\| \le \alpha e^{\omega^2 t}, t \ge 0, \varepsilon > 0.$

(P3). $\varepsilon^2 G'(t;\varepsilon) = e^{-t/2\varepsilon^2}C(t/\varepsilon) + \frac{1}{2}\Big[R(t;\varepsilon) - G(t;\varepsilon)\Big].$

(P4). If $t(\varepsilon) > 0$ for $\varepsilon > 0$ with $t(\varepsilon)/\varepsilon^2 \to \infty$ as $\varepsilon \to 0$, then for every $T > 0$,

$$\lim_{\varepsilon\to 0} \sup_{t(\varepsilon)\le t\le T} \|R(t;\varepsilon)x - S(t)x\| = 0, \quad \text{and} \quad \lim_{\varepsilon\to 0} \sup_{t(\varepsilon)\le t\le T} \|G(t;\varepsilon)x - S(t)x\| = 0,$$

uniformly for x in bounded subsets of X.

(P5). $\|e^{-t/2\varepsilon^2}C(t/\varepsilon)u_0(\varepsilon) + \frac{1}{2}R(t;\varepsilon)u_0(\varepsilon) + G(t;\varepsilon)\Big[\frac{1}{2}u_0(\varepsilon) + \varepsilon^2 u_1(\varepsilon)\Big] - S(t)w_0\|$

$$\le \alpha e^{\omega^2 t}\Big[\varepsilon^2(1+\omega^2 t)\|Aw_0\| + \|u_0(\varepsilon) - w_0\| + \varepsilon^2\|u_1(\varepsilon)\|\Big], \quad t \ge 0.$$

Now let $T > 0$ be fixed and consider for $t \in [0, T]$,

$$\begin{aligned} u(t;\varepsilon) - w(t) &= e^{-t/2\varepsilon^2}C(t/\varepsilon)u_0(\varepsilon) + \frac{1}{2}R(t;\varepsilon)u_0(\varepsilon) + G(t;\varepsilon)\Big[\frac{1}{2}u_0(\varepsilon) + \varepsilon^2 u_1(\varepsilon)\Big] \\ &\quad - S(t)w_0 + \int_0^t \Big[G(t-s;\varepsilon)\hat{f}(s;\varepsilon) - S(t-s)\hat{f}(s)\Big]ds. \end{aligned} \tag{2.7}$$

By (H4) and (P5), we can write (2.7) as

$$\begin{aligned} u(t;\varepsilon) - w(t) &= 0(\varepsilon, [0,T]) + \int_0^t \Big[G(t-s;\varepsilon)\hat{f}(s;\varepsilon) - S(t-s)\hat{f}(s)\Big]ds \\ &= 0(\varepsilon, [0,T]) + \int_0^t G(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon) - \hat{f}(s)\Big]ds \\ &\quad + \int_0^t \Big[G(t-s;\varepsilon) - S(t-s)\Big]\hat{f}(s)ds, \end{aligned} \tag{2.8}$$

where $0(\varepsilon, [0,T])$ satisfies

$$0(\varepsilon, [0,T]) \to 0 \ \text{ as } \ \varepsilon \to 0, \ \text{ uniformly for } \ t \in [0,T]. \tag{2.9}$$

Note that w is locally bounded and $f \in L^1([0,T], X)$, then $\hat{f} \in L^1([0,T], X)$. So from [5],

$$\int_0^t \Big[G(t-s;\varepsilon) - S(t-s)\Big]\hat{f}(s)ds = 0(\varepsilon, [0,T]), \ t \in [0,T]. \tag{2.10}$$

Next, we have

$$\begin{aligned} \int_0^t G(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon) - \hat{f}(s)\Big]ds &= \int_0^t G(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon) - \hat{f}(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]ds \\ &\quad - \int_0^t G(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)ds. \end{aligned} \tag{2.11}$$

Now from (P3),

$$\begin{aligned}
&\int_0^t G(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)ds\\
&= \varepsilon^2 G(0;\varepsilon)F(0)u(t;\varepsilon) - \varepsilon^2 G(t;\varepsilon)F(0)u_0(\varepsilon)\\
&\quad + \varepsilon^2 \int_0^t G'(t-s;\varepsilon)F(0)u(s;\varepsilon)ds\\
&= \varepsilon^2 G(0;\varepsilon)F(0)u(t;\varepsilon) - \varepsilon^2 G(t;\varepsilon)F(0)u_0(\varepsilon)\\
&\quad + \varepsilon^2 \int_0^t G'(t-s;\varepsilon)F(0)\Big[u(s;\varepsilon)-w(s)\Big]ds + \varepsilon^2 \int_0^t G'(t-s;\varepsilon)F(0)w(s)ds\\
&= \varepsilon^2 G(0;\varepsilon)F(0)\Big[u(t;\varepsilon)-w(t)\Big] + \varepsilon^2 G(0;\varepsilon)F(0)w(t)\\
&\quad - \varepsilon^2 G(t;\varepsilon)F(0)u_0(\varepsilon) + \varepsilon^2 \int_0^t G'(t-s;\varepsilon)F(0)\Big[u(s;\varepsilon)-w(s)\Big]ds\\
&\quad + \int_0^t \Big\{e^{-(t-s)/2\varepsilon^2} C((t-s)/\varepsilon) + \frac{1}{2}\Big[R(t-s;\varepsilon) - G(t-s;\varepsilon)\Big]\Big\} F(0)w(s)ds. \quad (2.12)
\end{aligned}$$

Observe that $w(s)$ is locally bounded, so use property (P4) with $t(\varepsilon) = \varepsilon$ to obtain for any $t, s \in [0,T]$ with $s < t$,

$$\Big[R(t-s;\varepsilon) - G(t-s;\varepsilon)\Big]F(0)w(s) \to 0, \ \ \varepsilon \to 0. \quad (2.13)$$

Hence the dominated convergence theorem can be used to prove that

$$\int_0^t \Big[R(t-s;\varepsilon) - G(t-s;\varepsilon)\Big]F(0)w(s)ds \to 0, \ \ \varepsilon \to 0, \quad (2.14)$$

uniformly for $t \in [0,T]$. Next, assume that $\varepsilon > 0$ is so small that $4\varepsilon\omega^2 \le 1$, then from (P1),

$$\begin{aligned}
&\int_0^t e^{-(t-s)/2\varepsilon^2}\|C((t-s)/\varepsilon)\|ds = \int_0^t e^{-s/2\varepsilon^2}\|C(s/\varepsilon)\|ds\\
&\le \alpha \int_0^t e^{-s/2\varepsilon^2 + \omega^2 s/\varepsilon} ds = \Big[2\alpha\varepsilon^2/(1-2\varepsilon\omega^2)\Big]\Big[1 - e^{(2\varepsilon\omega^2-1)t/2\varepsilon^2}\Big]\\
&\le 4\alpha\varepsilon^2 \to 0, \ \ \varepsilon \to 0, \quad (2.15)
\end{aligned}$$

uniformly for $t \in [0,T]$. Also observe that $w(\cdot)$ is locally bounded and $u_0(\varepsilon)$ has a limit as $\varepsilon \to 0$, then from (P2),

$$\varepsilon^2 G(0;\varepsilon)F(0)w(t), \ \ \varepsilon^2 G(t;\varepsilon)F(0)u_0(\varepsilon) \longrightarrow 0, \varepsilon \to 0,$$

uniformly for $t \in [0,T]$, and

$$\begin{aligned}
&\|\varepsilon^2 \int_0^t G'(t-s;\varepsilon)F(0)\Big[u(s;\varepsilon)-w(s)\Big]ds\|\\
&\qquad \le \alpha e^{\omega^2 T}\|F(0)\| \int_0^t \|u(s;\varepsilon) - w(s)\|ds.
\end{aligned}$$

Thus by (2.12), (2.14), (2.15), and property (P2), we obtain

$$\begin{aligned}\| \int_0^t G(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)ds - \varepsilon^2 G(0;\varepsilon)F(0)\Big[u(t;\varepsilon) - w(t)\Big] \| \\ \leq (\text{type 1}) + 0(\varepsilon,[0,T]),\end{aligned} \tag{2.16}$$

where (type 1) is of the form

$$(\text{constant}) \int_0^t \|u(s;\varepsilon) - w(s)\| ds. \tag{2.17}$$

Next we have

$$\begin{aligned}&\int_0^t G(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon) - \hat{f}(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]ds \\ &= \int_0^t G(t-s;\varepsilon)\Big\{\Big[f(s;\varepsilon) - f(s)\Big] + \int_0^s F(s-h)\Big[f(h;\varepsilon) - f(h)\Big]dh \\ &\quad + F(s)\Big[u_0(\varepsilon) - w_0\Big] - \int_0^s F'(s-h)\Big[u(h;\varepsilon) - w(h)\Big]dh + F(s)\varepsilon^2 u_1(\varepsilon) + \varepsilon^2 F'(s)u_0(\varepsilon) \\ &\quad - \Big[\varepsilon^2 F'(0) + F(0)\Big]\Big[u(s;\varepsilon) - w(s)\Big] - \varepsilon^2 F'(0)w(s) - \varepsilon^2 \int_0^s F''(s-h)\Big[u(h;\varepsilon) - w(h)\Big]dh \\ &\quad - \varepsilon^2 \int_0^s F''(s-h)w(h)dh\Big\}ds.\end{aligned}$$

Note that from (P2),

$$\begin{aligned}&\| \int_0^t G(t-s;\varepsilon)\int_0^s F(s-h)\Big[f(h;\varepsilon) - f(h)\Big]dhds \| \\ &\leq \alpha e^{\omega^2 T}\Big[\int_0^T \|F(s)\|ds\Big]\Big[\int_0^T \|f(s;\varepsilon) - f(s)\|ds\Big], \\ &\| \int_0^t G(t-s;\varepsilon)F(s)\Big[u_0(\varepsilon) - w_0\Big]ds \| \\ &\leq \alpha e^{\omega^2 T}\|u_0(\varepsilon) - w_0\| \int_0^T \|F(s)\|ds, \\ &\| \int_0^t G(t-s;\varepsilon)\int_0^s F'(s-h)\Big[u(h;\varepsilon) - w(h)\Big]dhds \| \\ &\leq \alpha e^{\omega^2 T}\Big[\int_0^T \|F'(s)\|ds\Big]\Big[\int_0^T \|u(s;\varepsilon) - w(s)\|ds\Big].\end{aligned}$$

Other terms can be treated similarly. So it is clear that with property (P2), hypotheses (H1) – (H5), and the fact that $w(\cdot)$ is locally bounded, we obtain

$$\| \int_0^t G(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon) - \hat{f}(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]ds \| \leq (\text{type 1}) + 0(\varepsilon,[0,T]). \tag{2.18}$$

Combine (2.8), (2.10), (2.11), (2.16), and (2.18), we get

$$\|(1+\varepsilon^2 G(0;\varepsilon)F(0))\Big[u(t;\varepsilon)-w(t)\Big]\| \le (\text{type } 1) + 0(\varepsilon,[0,T]), \tag{2.19}$$

Now assume $\varepsilon > 0$ is so small that $2\|\varepsilon^2 G(0;\varepsilon)F(0)\| < 1$, then

$$\|u(t;\varepsilon)-w(t)\| \le 0(\varepsilon,[0,T]) + (\text{constant})\int_0^t \|u(s;\varepsilon)-w(s)\|ds,\ t\in[0,T]. \tag{2.20}$$

So that the Gronwall's inequality ([9]) can be used to obtain

$$\|u(t;\varepsilon)-w(t)\| \le 0(\varepsilon,[0,T]),\ \ t\in[0,T]. \tag{2.21}$$

This proves the theorem. □

Finally, we briefly indicate its applications in viscoelasticity. Let us consider

$$\begin{aligned} \rho u_{tt}(t;\rho)+\alpha u_t(t;\rho) &= \Delta u(t;\rho)+\int_0^t K(t-s)\Delta u(s;\rho)ds + f(t;\rho), \quad t\ge 0, \\ u(0;\rho) &= u_0(\rho),\ \ u_t(0;\rho)=u_1(\rho), \end{aligned} \tag{2.22}$$

in $L^2(\Omega)$, where u is the displacement, ρ is the density per unit area, and α is the coefficient of viscosity of the medium. With appropriate boundary conditions the Laplacian operator Δ in Eq.(2.22) generates a strongly continuous cosine family and a strongly continuous semigroup. So with some convergence conditions on initial data and $f(t;\varepsilon)$ and smoothness conditions on $K(\cdot)$, Theorems 2.1 can be used to show that when density $\rho\to 0$, solutions of (2.22) will converge to solutions of the "limiting" heat equation

$$\alpha w_t(t) = \Delta w(t) + \int_0^t K(t-s)\Delta w(s)ds + f(t),\ \ t\ge 0,\ \ w(0)=w_0. \tag{2.23}$$

Details are omitted here. This result also relates to a concept called "change the type" (from hyperbolic to parabolic).

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