SINGULAR PERTURBATIONS IN A NON-LINEAR VISCOELASTICITY

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Abstract

A non-linear equation in viscoelasticity of the form

$$\begin{aligned}
\rho u^\rho_{tt}(t,x) &= \phi(u^\rho_x(t,x))_x + \int_{-\infty}^t F(t-s)\phi(u^\rho_x(s,x))_x ds + \rho g(t,x) + f(x),\ t\geq 0,\ x\in[0,1], && (0.1)\\
u^\rho(t,0) &= u^\rho(t,1) = 0,\ t\geq 0, && (0.2)\\
u^\rho(s,x) &= v^\rho(s,x),\ s\leq 0,\ x\in[0,1], && (0.3)
\end{aligned}$$

(where ϕ is non-linear) is studied when the density ρ of the material goes to zero. It will be shown that when $\rho\downarrow 0$, solutions u^ρ of the dynamical system (0.1)-(0.3) approach the unique solution w (which is independent of t) of the steady state obtained from (0.1)-(0.3) with $\rho=0$. Moreover, the rate of convergence in ρ is obtained to be $\|u^\rho - w\|_{L^2}\leq K\sqrt{\rho}$ and $\|u^\rho_x - w_x\|_{L^2}\leq K\sqrt{\rho}$ for some constant K independent of ρ.

1 INTRODUCTION.

Let us begin with the following quasi-static approximation studied in MacCamy [11],

$$u_{tt}(t) = -A(0)g(u(t)) - \int_0^t A'(t-s)g(u(s))ds + F(t), \tag{1.1}$$

and

$$0 = -A(0)g(w(t)) - \int_0^t A'(t-s)g(w(s))ds + F(t). \tag{1.2}$$

Here $A(t)$ is a bounded and linear operator and g is a non-linear and unbounded operator in a Hilbert space. It is shown in [11] that if $F(t)$ approaches a constant vector $F(\infty)$ as

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$t \to \infty$, then, under appropriate conditions, one has

$$g(u(t)) \to A(\infty)^{-1}F(\infty) \text{ weakly in } H, \text{ as } t \to \infty, \quad (1.3)$$
$$g(w(t)) \to A(\infty)^{-1}F(\infty) \quad \text{ in } H, \text{ as } t \to \infty, \quad (1.4)$$

where u and w are solutions of (1.1) and (1.2) respectively. This result motivates the procedure of using the quasi-static approximation in viscoelasticity, which drops the "acceleration" term u_{tt} when t is large. That is, use w to approximate u.

Now, let us look at the following non-linear equation in viscoelasticity,

$$\begin{aligned} \rho u^\rho_{tt}(t,x) &= \phi(u^\rho_x(t,x))_x + \int_{-\infty}^t F(t-s)\phi(u^\rho_x(s,x))_x ds \\ &\quad +\rho g(t,x) + f(x), \quad t \geq 0,\ x \in [0,1], \quad (1.5) \\ u^\rho(t,0) &= u^\rho(t,1) = 0,\ t \geq 0; \quad u^\rho(s,x) = v^\rho(s,x),\ \ s \leq 0,\ x \in [0,1], \quad (1.6) \end{aligned}$$

which can be found in e.g., Dafermos and Nohel [4] and MacCamy [13]. Here u is the displacement, ρg is the body force, f is the external force, and ρ is the density of the material. Same as in MacCamy [13], we assume that ϕ on $\Re$ is non-linear, $\phi(0) = 0$, and there is a constant $c_0 > 0$ such that $\phi' \geq c_0$ on $\Re$.

For Eq.(1.5)-(1.6), we propose the singular perturbation problem in the following sense: show that when $\rho \downarrow 0$, the solutions of (1.5)-(1.6) approach the solutions of the equation obtained from (1.5)-(1.6) with $\rho = 0$. It will be shown that the solution of (1.5)-(1.6) with $\rho = 0$ exists uniquely and is independent of t, i.e., in static-state. Thus, this singular perturbation can also be regarded as a quasi-static approximation.

When ϕ is linear, (1.5)-(1.6) is studied in Grimmer and Liu [6], where linearity is used to subtract the solution w of (1.5)-(1.6) with $\rho = 0$ from the solutions u^ρ of (1.5)-(1.6). Then an equation for $Q^\rho \equiv u^\rho - w$ is formulated and the method of energy estimate is employed to show that $(u^\rho - w =)\, Q^\rho \to 0$ as $\rho \to 0$.

When ϕ is non-linear but $f = 0$, it is shown in [6] that the solution w of (1.5)-(1.6) with $\rho = 0$ is $w = 0$. Thus the equation for $Q^\rho \equiv u^\rho - w = u^\rho$ is the same as Eq.(1.5)-(1.6) (with $f = 0$). Therefore, it is indicated in [6] that the energy estimate method can be modified to show that $(u^\rho - w = u^\rho =)\, Q^\rho \to 0$ as $\rho \to 0$.

Now, in this paper, we look at the case where ϕ is non-linear and $f \neq 0$. It will be seen that this case is complicated than the previous cases. For example, the equation for

$Q^\rho \equiv u^\rho - w$ also involves w. However, after some trials and errors, we found an appropriate energy function for Q^ρ so that the method of the energy estimate used in [6] can also be extended here to show that $(u^\rho - w =)\ Q^\rho \to 0$ as $\rho \to 0$. Moreover, the rate of convergence in ρ is obtained to be $\|u^\rho - w\|_{L^2} \leq K\sqrt{\rho}$ and $\|u_x^\rho - w_x\|_{L^2} \leq K\sqrt{\rho}$ for some constant K independent of ρ, as a by-product of our energy estimate in this paper. (The rate of convergence was not discovered in [6].)

Related studies of singular perturbations can be found in, for example, Chow and Lu [1], Fattorini [5], Hale and Raugel [8], Grimmer and Liu [6], and Liu [9, 10].

2 SINGULAR PERTURBATIONS.

Note that the existence and uniqueness of solutions of Eq.(1.5)-(1.6) (with $\rho > 0$) were obtained in [4, 7, 12, 13], and we are only interested in singular perturbations in this paper, so we will assume that Eq.(1.5)-(1.6) (with $\rho > 0$) has a unique solution u^ρ for every $\rho > 0$. Also note that we first assume that the "history" v^ρ satisfies Eq.(1.5) on $\Re^-$. Then we will see that if v^ρ is only specified on $\Re^-$ (may not satisfy Eq.(1.5)), then with essentially the same proof, we can obtain the similar results.

Now we can state and prove our main results with the following hypothesis:

(H). $1 + \hat{F}(\lambda) \neq 0$ for $Re\lambda \geq 0$. F and $F' \in L^1(\Re^+)$. $F = 0$ on $\Re^-$. $f \in C[0,1]$. $\|v_t^\rho(s,\cdot)\|_{L^2}$ and $\|g(-s)\|_{L^2}$ are bounded for $s \leq 0$.

Here $\hat{F}$ is the Laplace transform of F, and $L^2 = L^2[0,T]$.

Theorem 2.1. Assume that the hypothesis (H) is satisfied. Then there is a unique w, which is independent of t, such that

$$0 = \phi(w_x(x))_x + \int_{-\infty}^t F(t-s)\phi(w_x(x))_x ds + f(x),\ t \in \Re,\ x \in [0,1], \tag{2.1}$$

$$w(0) = w(1) = 0. \tag{2.2}$$

(This equation is obtained from (1.5)-(1.6) with $\rho = 0$.)

Proof. Similar to [6], we let R be the function such that $R(s) = 0,\ s \leq 0$ and

$$R(t) = -F(t) - \int_0^t R(t-s)F(s)ds, \quad t \geq 0, \tag{2.3}$$

whose existence is studied in, e.g., [2, 3, 7]. Note that (2.3) can be written as

$$(\delta + R) * (\delta + F) = \delta, \tag{2.4}$$

where

$$R * F(t) = \int_{-\infty}^{t} R(t-s)F(s)ds \;\text{ and }\; \delta * H = H. \tag{2.5}$$

Now, write (1.5) with $\rho = 0$ as

$$-f(x) = (\delta + F) * \phi(u_x(t,x))_x. \tag{2.6}$$

This implies

$$\begin{aligned} \phi(u_x(t,x))_x &= -(\delta + R) * f(x) = -\Big[1 + \int_0^\infty R(s)ds\Big] f(x) \\ &= -\Big[1 + \int_0^\infty F(s)ds\Big]^{-1} f(x) \overset{\text{def}}{=} f_0(x). \end{aligned} \tag{2.7}$$

Thus we have

$$\phi(u_x(t,x)) = \int_0^x f_0(r)dr + C, \tag{2.8}$$

$$u_x(t,x) = \phi^{-1}\Big(\int_0^x f_0(r)dr + C\Big). \tag{2.9}$$

Therefore, the solution takes the following form

$$w(x) \overset{\text{def}}{=} u(t,x) = \int_0^x \phi^{-1}\Big(\int_0^s f_0(r)dr + C\Big)ds + C_1. \tag{2.10}$$

Taking into account of the boundary condition (1.6), we see that $C_1 = 0$ and that we only need to verify that there is a unique constant C such that

$$\int_0^1 \phi^{-1}\Big(\int_0^s f_0(r)dr + C\Big)ds = 0. \tag{2.11}$$

For this purpose, we first note that since $\phi' \geq c_0 > 0$ on $\Re$, one has $\phi^{-1}(-\infty) = -\infty$ and $\phi^{-1}(\infty) = \infty$. Thus there exists at least one C such that (2.11) is true.

Next, taking a derivative in C of the function

$$G(C) \equiv \int_0^1 \phi^{-1}\Big(\int_0^s f_0(r)dr + C\Big)ds, \tag{2.12}$$

one gets

$$\frac{1}{c_0} \geq G'(C) = \int_0^1 \frac{1}{\phi'\Big(\phi^{-1}\Big(\int_0^s f_0(r)dr + C\Big)\Big)} ds > 0. \tag{2.13}$$

Therefore $G(C)$ is strictly increasing in C. Hence, there exists a unique C such that (2.11) is true. $\square$

Theorem 2.2. Assume that the hypothesis (H) is satisfied and that Eq.(1.5)-(1.6) has a unique solution u^ρ (on $\Re$) for $\rho > 0$ (i.e., v^ρ satisfies Eq.(1.5)-(1.6) on $\Re^-$). Let w be the unique solution of (1.5)-(1.6) with $\rho = 0$ (from Theorem 2.1). For $T > 0$ fixed and $t \in [0, T],\ x \in [0, 1]$, define $Q^\rho(t, x) \equiv u^\rho(t, x) - w(x)$ and

$$E(t;\rho) \equiv \int_0^1 \Big[Q_t^\rho(t,x)\Big]^2 dx + \frac{2}{\rho}\int_0^1 \int_0^{Q_x^\rho(t,x)} \Big[\phi(r + w_x(x)) - \phi(w_x(x))\Big] dr dx. \tag{2.14}$$

If there exists a constant K_0 independent of ρ such that $E(0, \rho) \leq K_0,\ \rho > 0$, then as $\rho \to 0$, we have $u^\rho(t, \cdot) \to w(\cdot)$ and $u_x^\rho(t, \cdot) \to w_x(\cdot)$ in $C([0, T], L^2[0, T])$. Moreover, there exists a constant K independent of ρ such that

$$\|u^\rho(t,\cdot) - w(\cdot)\|_{L^2} \leq K\sqrt{\rho}, \quad \|u_x^\rho(t,\cdot) - w_x(\cdot)\|_{L^2} \leq K\sqrt{\rho}, \quad t \in [0,T],\ \rho > 0. \tag{2.15}$$

Remark 2.1. $E(0, \rho)$ is bounded when, for example, $v_t^\rho(0, x)$ is bounded and $Q_x^\rho(0, x) = 0$ (i.e., $v_x^\rho(0, x) = w_x(x)$), independently of ρ.

Proof of Theorem 2.2. We first verify that

$$\int_0^t [\phi(r + s) - \phi(s)]dr \geq \frac{c_0}{2} t^2, \quad t, s \in \Re. \tag{2.16}$$

For this purpose let us use the Mean Value Theorem and get

$$\int_0^t [\phi(r + s) - \phi(s)]dr = \int_0^t \phi'(\xi) r dr. \tag{2.17}$$

If $t > 0$, then $r \geq 0$ and

$$\int_0^t \phi'(\xi) r dr \geq c_0 \int_0^t r dr = \frac{c_0}{2} t^2. \tag{2.18}$$

If $t < 0$, then $r \leq 0$ and

$$\begin{aligned} \int_0^t \phi'(\xi) r dr &= \int_t^0 \phi'(\xi)(-r) dr \\ &\geq c_0 \int_t^0 (-r) dr = \frac{c_0}{2} t^2. \end{aligned} \tag{2.19}$$

Next, we show that for the $E(t;\rho)$ defined by (2.14) with $E(0;\rho) \leq K_0$, there exists a constant K_1 independent of ρ such that $E(t;\rho) \leq K_1,\ \rho > 0,\ t \in [0,T]$.

For this end we first note that from (2.16), one has

$$\int_0^1 \int_0^{Q_x^\rho(t,x)} [\phi(r + w_x(x)) - \phi(w_x(x))]drdx \geq \frac{c_0}{2} \int_0^1 \Big[Q_x^\rho(t,x)\Big]^2 dx \geq 0. \tag{2.20}$$

Then, observe that since we assumed that u^ρ satisfies Eq.(1.5) on $\Re$, the equation for $Q^\rho(t,x) \equiv u^\rho(t,x) - w(x)$ is

$$\begin{aligned} \rho Q_{tt}^\rho(t,x) &= \Big[\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))\Big]_x \\ &\quad + \int_{-\infty}^t F(t-s)\Big[\phi(Q_x^\rho(s,x) + w_x(x)) - \phi(w_x(x))\Big]_x ds + \rho g(t,x) \end{aligned} \tag{2.21}$$

for $t \in \Re$. Using (2.5), this can be written as

$$\rho\Big(Q_{tt}^\rho(t,x) - g(t,x)\Big) = (\delta + F) * \Big[\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))\Big]_x,\quad t \in \Re. \tag{2.22}$$

Now, note that from [6, 14] one has $R(\infty) = 0$. Hence,

$$\begin{aligned} &\Big[\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))\Big]_x = \rho(\delta + R) * \Big(Q_{tt}^\rho(t,x) - g(t,x)\Big) \\ &= \rho\Big(Q_{tt}^\rho(t,x) - g(t,x) + \int_{-\infty}^t R(t-s)\Big[Q_{tt}^\rho(s,x) - g(s,x)\Big]ds\Big) \\ &= \rho\Big(Q_{tt}^\rho(t,x) - g(t,x) + R(0)Q_t^\rho(t,x) + \int_{-\infty}^t R'(t-s)Q_t^\rho(s,x)ds \\ &\quad - \int_{-\infty}^t R(t-s)g(s,x)ds\Big). \end{aligned} \tag{2.23}$$

Next, take a derivative of $E(t;\rho)$ in t and use the boundary condition (1.6) to get

$$\begin{aligned} \frac{d}{dt}E(t;\rho) &= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)dx + \frac{2}{\rho}\int_0^1 \Big[\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))\Big]Q_{xt}^\rho(t,x)dx \\ &= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)dx - \frac{2}{\rho}\int_0^1 \Big[\phi(Q_x^\rho(t,x) + w_x(x)) - \phi(w_x(x))\Big]_x Q_t^\rho(t,x)dx. \end{aligned}$$

Then, replace (2.23) into it to obtain

$$\frac{d}{dt}E(t;\rho) = 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)dx - 2\int_0^1 \Big(Q_{tt}^\rho(t,x) - g(t,x)$$

$$
\begin{aligned}
&+R(0)Q_t^\rho(t,x)+\int_{-\infty}^t R'(t-s)Q_t^\rho(s,x)ds-\int_{-\infty}^t R(t-s)g(s,x)ds\Big)Q_t^\rho(t,x)dx\\
=\;& 2\int_0^1\Big(g(t,x)-R(0)Q_t^\rho(t,x)-\int_{-\infty}^t R'(t-s)Q_t^\rho(s,x)ds\\
&+\int_{-\infty}^t R(t-s)g(s,x)ds\Big)Q_t^\rho(t,x)dx\\
\le\;& \|g(t,\cdot)\|_{L^2}^2+\Big(2+2|R(0)|\Big)\|Q_t^\rho(t,\cdot)\|_{L^2}^2\\
&+\int_{-\infty}^t |R'(t-s)|\Big[\|Q_t^\rho(s,\cdot)\|_{L^2}^2+\|Q_t^\rho(t,\cdot)\|_{L^2}^2\Big]ds\\
&+\int_0^1\Big[\int_{-\infty}^t |R(t-s)g(s,x)|ds\Big]^2dx\\
\le\;& \Big(2+2|R(0)|+\int_0^\infty |R'(s)|ds\Big)\|Q_t^\rho(t,\cdot)\|_{L^2}^2\\
&+\int_0^t |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2ds\\
&+\|g(t,\cdot)\|_{L^2}^2+\int_{-\infty}^0 |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2ds+\int_0^1\Big[\int_{-\infty}^t |R(t-s)g(s,x)|ds\Big]^2dx.
\end{aligned}
$$

Now, note that $\|Q_t^\rho(t,\cdot)\|_{L^2}^2\le E(t;\rho)$ by (2.20). Then from above one gets

$$
\begin{aligned}
\frac{d}{dt}E(t;\rho)\;\le\;& \Big(2+2|R(0)|+\int_0^\infty |R'(s)|ds\Big)E(t;\rho)\\
&+\int_0^t |R'(t-s)|E(s;\rho)ds\\
&+\|g(t,\cdot)\|_{L^2}^2+\int_{-\infty}^0 |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2ds+\int_0^1\Big[\int_{-\infty}^t |R(t-s)g(s,x)|ds\Big]^2dx\\
\le\;& HE(t;\rho)+\int_0^t |R'(t-s)|E(s;\rho)ds+P, \qquad (2.24)
\end{aligned}
$$

where H and P are constants defined in a obvious way.

Similar to [6], we can use the standard arguments in differential inequality to obtain a constant K_1 independent of ρ such that $E(t;\rho)\le K_1,\ t\in[0,T],\ \rho>0$. Therefore, (2.20) implies

$$
\frac{c_0}{\rho}\int_0^1\Big[Q_x^\rho(t,x)\Big]^2dx\le E(t;\rho)\le K_1,\quad t\in[0,T],\ \rho>0. \qquad (2.25)
$$

Now, note that the boundary condition in (1.6) implies

$$
\|Q^\rho(t,\cdot)\|_{L^2}\le\|Q_x^\rho(t,\cdot)\|_{L^2}. \qquad (2.26)
$$

Thus we can let $K \equiv \sqrt{K_1/c_0}$ and obtain

$$\|Q^\rho(t,\cdot)\|_{L^2} \le \|Q^\rho_x(t,\cdot)\|_{L^2} \le K\sqrt{\rho}, \quad t \in [0,T],\ \rho > 0. \tag{2.27}$$

This proves the Theorem. $\square$

Remark 2.2. Here, the proof of $Q^\rho(t,x) \to 0$ as $\rho \to 0$ is different from [6], and is short and direct, and can also provide the rate of convergence in ρ.

In the following, we will verify that if v^ρ is only specified on $\Re^-$ and may not satisfy Eq.(1.5), then we can still get the similar results. Because now, (2.21) becomes

$$\begin{aligned} \rho Q^\rho_{tt}(t,x) \;=\;& \Big[\phi(Q^\rho_x(t,x)+w_x(x)) - \phi(w_x(x))\Big]_x \\ &+\int_0^t F(t-s)\Big[\phi(Q^\rho_x(s,x)+w_x(x)) - \phi(w_x(x))\Big]_x ds \\ &+\int_{-\infty}^0 F(t-s)\Big[\phi(Q^\rho_x(s,x)+w_x(x)) - \phi(w_x(x))\Big]_x ds \\ &+\rho g(t,x), \quad t \ge 0. \end{aligned} \tag{2.28}$$

And hence, (2.22) becomes

$$\begin{aligned} \rho\Big(Q^\rho_{tt}(t,x) - g(t,x)\Big) \;=\;& (\delta+F)\hat{*}\Big[\phi(Q^\rho_x(t,x)+w_x(x)) - \phi(w_x(x))\Big]_x \\ &+\int_{-\infty}^0 F(t-s)\Big[\phi(Q^\rho_x(s,x)+w_x(x)) - \phi(w_x(x))\Big]_x ds \end{aligned} \tag{2.29}$$

where the integration in $\hat{*}$ is from 0 to t. Therefore (2.23) becomes

$$\begin{aligned} &\Big[\phi(Q^\rho_x(t,x)+w_x(x)) - \phi(w_x(x))\Big]_x = (\delta+R)\hat{*}\Big\{\rho\Big(Q^\rho_{tt}(t,x) - g(t,x)\Big) \\ &\quad -\int_{-\infty}^0 F(t-s)\Big[\phi(Q^\rho_x(s,x)+w_x(x)) - \phi(w_x(x))\Big]_x ds\Big\} \\ &= \rho\Big(Q^\rho_{tt}(t,x) - g(t,x) + \int_0^t R(t-s)\Big[Q^\rho_{tt}(s,x) - g(s,x)\Big]ds\Big) \\ &\quad -(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\Big[\phi(Q^\rho_x(s,x)+w_x(x)) - \phi(w_x(x))\Big]_x ds \\ &= \rho\Big(Q^\rho_{tt}(t,x) - g(t,x) + R(0)Q^\rho_t(t,x) - R(t)Q^\rho_t(0,x) \\ &\quad +\int_0^t R'(t-s)Q^\rho_t(s,x)ds - \int_0^t R(t-s)g(s,x)ds\Big) \\ &\quad -(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\Big[\phi(Q^\rho_x(s,x)+w_x(x)) - \phi(w_x(x))\Big]_x ds. \end{aligned} \tag{2.30}$$

Thus, (2.24) will be changed to

$$
\begin{aligned}
\frac{d}{dt}E(t;\rho) &= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)dx - \frac{2}{\rho}\int_0^1 \Big[\phi(Q_x^\rho(t,x)+w_x(x))-\phi(w_x(x))\Big]_x Q_t^\rho(t,x)dx \\
&= 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)dx - 2\int_0^1 \Big(Q_{tt}^\rho(t,x)-g(t,x)+R(0)Q_t^\rho(t,x) \\
&\quad -R(t)Q_t^\rho(0,x)+\int_0^t R'(t-s)Q_t^\rho(s,x)ds-\int_0^t R(t-s)g(s,x)ds\Big)Q_t^\rho(t,x)dx \\
&\quad +\frac{2}{\rho}\int_0^1 \Big\{(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\Big[\phi(Q_x^\rho(s,x)+w_x(x)) \\
&\quad -\phi(w_x(x))\Big]_x ds\Big\}Q_t^\rho(t,x)dx \\
&= 2\int_0^1 \Big(g(t,x)+R(t)Q_t^\rho(0,x)-R(0)Q_t^\rho(t,x) \\
&\quad -\int_0^t R'(t-s)Q_t^\rho(s,x)ds+\int_0^t R(t-s)g(s,x)ds\Big)Q_t^\rho(t,x)dx \\
&\quad +2\int_0^1 \Big\{\frac{1}{\rho}(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\Big[\phi(Q_x^\rho(s,x)+w_x(x)) \\
&\quad -\phi(w_x(x))\Big]_x ds\Big\}Q_t^\rho(t,x)dx \\
&\le \|g(t,\cdot)+R(t)Q_t^\rho(0,\cdot)\|_{L^2}^2+\Big(3+2|R(0)|\Big)\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \\
&\quad +\int_0^t |R'(t-s)|\Big[\|Q_t^\rho(s,\cdot)\|_{L^2}^2+\|Q_t^\rho(t,\cdot)\|_{L^2}^2\Big]ds \\
&\quad +\int_0^1 \Big[\int_0^t |R(t-s)g(s,x)|ds\Big]^2 dx \\
&\quad +\int_0^1 \Big\{\frac{1}{\rho}(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\Big[\phi(Q_x^\rho(s,x)+w_x(x))-\phi(w_x(x))\Big]_x ds\Big\}^2 dx \\
&\le \Big(3+2|R(0)|+\int_0^\infty |R'(s)|ds\Big)\|Q_t^\rho(t,\cdot)\|_{L^2}^2 \\
&\quad +\int_0^t |R'(t-s)|\|Q_t^\rho(s,\cdot)\|_{L^2}^2 ds \\
&\quad +\|g(t,\cdot)+R(t)Q_t^\rho(0,\cdot)\|_{L^2}^2+\int_0^1 \Big[\int_0^t |R(t-s)g(s,x)|ds\Big]^2 dx \\
&\quad +\int_0^1 \Big\{(\delta+R)\hat{*}\int_{-\infty}^0 F(t-s)\frac{1}{\rho}\Big[\phi(v_x^\rho(s,x))-\phi(w_x(x))\Big]_x ds\Big\}^2 dx \\
&\le \hat{H}E(t;\rho)+\int_0^t |R'(t-s)|E(s;\rho)ds+\hat{P}. \qquad (2.31)
\end{aligned}
$$

Now, it is clear that we have the following result, which is similar to Theorem 2.2:

Theorem 2.3. Assume that the hypothesis (H) is satisfied and that Eq.(1.5)-(1.6) has a unique solution u^ρ (on $\Re^+$) for $\rho > 0$ (i.e., v^ρ is only specified on $\Re^-$ and may not satisfy Eq.(1.5)-(1.6) on $\Re^-$). Let w be the unique solution of (1.5)-(1.6) with $\rho = 0$ (from Theorem 2.1). Assume further that for some constant C independent of ρ,

$$\frac{1}{\rho}\left|\Big[\phi(v^\rho_x(s,x)) - \phi(w_x(x))\Big]_x\right| \le C, \quad s \le 0,\ x \in [0,1],\ \rho > 0. \tag{2.32}$$

If there exists a constant K_0 independent of ρ such that $E(0,\rho) \le K_0,\ \rho > 0$, then as $\rho \to 0$, we have $u^\rho(t,\cdot) \to w(\cdot)$ and $u^\rho_x(t,\cdot) \to w_x(\cdot)$ in $C([0,T], L^2[0,T])$. Moreover, there exists a constant K independent of ρ such that

$$\|u^\rho(t,\cdot) - w(\cdot)\|_{L^2} \le K\sqrt{\rho}, \ \ \|u^\rho_x(t,\cdot) - w_x(\cdot)\|_{L^2} \le K\sqrt{\rho}, \quad t \in [0,T],\ \rho > 0. \tag{2.33}$$

Remark 2.3. (2.32) is satisfied if, for example, $v^\rho_x(s,x) = w_x(x),\ s \le 0,\ x \in [0,1],\ \rho > 0$.

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