A Collocation Expansion Approach to the Boundary Integral Method

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ABSTRACT

Over the past years, the Boundary Integral Method has shown itself to be an increasingly useful formulation for the numerical solution of varied problems in engineering and physics. Virtually all applications are made using the Boundary Element Method, where the surface and boundary variables are discretised into elements, typically of constant, linear, or sometimes quadratic variation over each element. We outline here a new method, where we separate the surface discretisation from the boundary variable approximations, and use an expansion method to find the unknowns. An outline of this collocation expansion method will be given, as well as some numerical comparisons with the more traditional Boundary Element Methods, for 2D potential problems. We will also discuss the relative merits of the methods.

INTRODUCTION

There has developed a considerable quantity of literature concerning the Boundary Element Method in recent years. Not only are there numerous entries in computational journals, but various publications have been made purely on the Boundary Element Method, among them Brebbia et. al. [4], Banerjee and Butler [1], the "Boundary Elements" series including Brebbia et. al. [3], and the "Developments in Boundary Elements" series including Banerjee and Watson [2]. Most of the work in these texts involves development of the Boundary Element Method and applications of it to various problems. However, virtually all of these applications make use of the standard constant, linear, and sometimes quadratic or higher element discretisations of the surface and the boundary values, both known and un-

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Figure 1: Number of unknowns vs. maximum error in solving equation (1) using various methods

known. As justification for an alternative approach to solving the Boundary Integral Equations, consider the solution of the Fredholm integral equation of the 2nd kind

$$x(s) = e^s - \frac{e^{1+s}-1}{1+s} + \int_0^1 e^{st}x(t)\,dt, \quad \text{with solution} \quad x(s) = e^s. \qquad (1)$$

Figure 1 shows the results for constant, linear and quadratic discretisations of the unknown $x(s)$ of equation (1), as well as a solution using the Fast Galerkin expansion method as described in Delves and Mohammed [6] and a collocation expansion technique, collocating at the zeros of the appropriate Chebychev polynomials. As we can see, and this is typical of most problems, expansion methods can be very effective. The basis of this paper will be the solution of the Boundary Integral Equations using a collocation based expansion method. We use a collocation rather than a Galerkin method to form our system of equations because the Galerkin method with the boundary integral kernels in 2D lead to double integrals with a line singularity along a diagonal. The additional computation required in this case is prohibitive, and in any case, a collocation expansion method is often as good, as in Figure 1.

THE COLLOCATION EXPANSION METHOD

For this work, we will restrict ourselves to 2D potential problems, which have the boundary integral formulation for a region Ω with surface Γ

$$c(\mathbf{r_0})\phi(\mathbf{r_0}) + \int_\Gamma \phi(\mathbf{r})\frac{\partial G}{\partial n}(\mathbf{r_0},\mathbf{r})\,ds(\mathbf{r}) = \int_\Gamma G(\mathbf{r_0},\mathbf{r})\frac{\partial \phi}{\partial n}(\mathbf{r})\,ds(\mathbf{r}), \qquad (2)$$

where $G(\mathbf{r_0},\mathbf{r}) = -\ln(|\mathbf{r_0},\mathbf{r}|)/(2\pi)$ and c depends on whether $\mathbf{r_0}$ is in Ω, or on Γ, and the smoothness of the curve at $\mathbf{r_0}$. Assume that the surface Γ is split into N curves, such that each curve is smooth apart from possibly at the endpoints, and let each curve be $\{C_i\}_{i-1}^N$ such that on C_i: $x = f_i(t)$, $y = g_i(t)$, $t : 0 \to s_i$, where s_i is the arc length of the curve. We also assume that on each C_i, one of either ϕ or $\partial\phi/\partial n$ is known. Then, equation (2) can be represented as

$$c\phi(\mathbf{r_0}) + \sum_{i=1}^N \int_0^{s_i} \phi(\mathbf{r})\frac{\partial G}{\partial n}(\mathbf{r_0},\mathbf{r})\sqrt{\left(\frac{df_i}{dt}\right)^2 + \left(\frac{dg_i}{dt}\right)^2}\,dt = \qquad (3)$$

$$\sum_{i=1}^N \int_0^{s_i} G(\mathbf{r_0},\mathbf{r})\frac{\partial \phi}{\partial n}(\mathbf{r})\sqrt{\left(\frac{df_i}{dt}\right)^2 + \left(\frac{dg_i}{dt}\right)^2}\,dt, \quad \text{where} \quad \mathbf{r} = (f_i(t), g_i(t)).$$

Since discontinuities cause significant difficulties for expansion method approximations, we assume N separate expansions, each of length m_i, one on each of the N curves, for the unknown on that curve. This gives

$$\int_0^{s_i} \chi(\mathbf{r}) K(\mathbf{r_0}, \mathbf{r})\, dt = \sum_{j=1}^{m_i} \alpha_{ij} \int_{-1}^{1} T_{j-1}(s) K(\mathbf{r_0}(s), \mathbf{r}(s)) \frac{s_i}{2}\, ds, \qquad (4)$$

where $\chi(\mathbf{r})$ is the unknown, $K(\mathbf{r_0}, \mathbf{r})$ is the appropriate kernel, s is the transform of t from $[0, s_i]$ onto $[-1, 1]$, $\{\alpha_{ij}\}_{j=1}^{m_i}$ are the expansion coefficients, and $T_{j-1}(s)$ is the (j-1)st Chebychev polynomial. The equivalent integral involving the known for the same i'th curve requires no further transformation.

Equation (3), using the result of equation (4), can be solved for the $M = \sum_{i=1}^{N} m_i$ unknown expansion coefficients by taking collocation points on the surface, placing m_i on the i'th curve. This will lead to an $M \times M$ system, which can be solved by Gauss elimination. The collocation points can be chosen as the zeros of the appropriate Chebychev polynomial, transferred onto $[0, s_i]$. With these collocation points, we immediately have $c = 1/2$, since away from the endpoints the curves are smooth.

Before discussing the implementation of the method, the curve representation and method of integration need to be outlined.

Arc length cubic splines

We make the functions (f_i, g_i) representing the curves arc length cubic splines (see de Boor [5] for theory and Kucera [8] for an application). If we are given a set of points $\{(x_i, y_i)\}_{i=1}^{L}$ for a curve, and the gradients of its endpoints, we can find the arc lengths of each point relative to the first, and construct clamped cubic splines for both x and y based upon the arc lengths $\{s_i\}_{i=1}^{L}$. Then, given a value of s in the appropriate range, x and y values on the curve can be found.

The arc lengths are found as follows: construct approximate s_i as zero for the first point, and the distance between points for the rest. Using these approximate s_i, calculate the splines, and from them calculate the arc lengths of the points on this curve. Using the new s_i, calculate new splines, and continue the process until the s_i converge. Typically, this process is only required a few times before a high degree of convergence is obtained.

Methods of integration

For all integrals required, we use the routine DQAGS from the QUADPACK numerical integration package of Piessens et. al. [9]. This routine is an adaptive rule that uses non-linear extrapolation in such a way that it

can deal with integrable singularities when thay are placed at an endpoint of the interval. Thus, integrals over curves including a singularity are split into two integrals at that singularity. The routine uses a 21 point Kronrod rule as its base quadrature, which has a 10 point Gauss-Legendre rule embedded in it, which is used for error estimation.

While the routine DQAGS gives reliable answers, it can tend to be computationally expensive. Further work is required to develop more efficient integration procedures for this method. We have already tried an adaptive rule using the IMT rule for sub-regions with the singularity at an endpoint, and the Kronrod rule on non-singular sub-regions. The IMT rule, as described in Iri et. al. [7], uses an exponential transform to remove endpoint singularities. Unfortunately, this leads to quadrature points so close to the singularity that double precision FORTRAN is not enough to eliminate round-off errors before reasonably accurate solutions are reached. For this problem, the IMT rule is not practical.

Implementation of the method

The input data required for this method is mainly points for the splines representing the curves. We use the standard convention of increasing arc length in an anti-clockwise direction giving an outward normal. For generality, known boundary values will be given at these points, and splines produced for these boundary values. Thus, enough points are required for accuracy in both the surface and boundary value splines. The number of expansion coefficients for a particular problem lead to the position of the collocation points, and the various integrals required are found as described earlier. All calculations are done in double precision FORTRAN, and integrals are done with a requested relative error of 5×10^{-6}. These results can be immediately added to the right hand side of the system if using known boundary values, or placed in the matrix of the left hand side if using Chebychev polynomials. Once set up, the system is solved, and the solution is split into the expansions on the various curves of the surface. Potential at internal points can then be calculated in the usual manner.

RESULTS

From now on, we will describe the methods by the names BICEM for the Boundary Integral Collocation Expansion Method, and BEM for the various Boundary Element Methods. The constant, linear and quadratic BEMs are as standard in the literature, with the exception that integrals are performed adaptively with the requested relative error estimate of 5×10^{-6} as in the BICEM, as opposed to the standard of using a set quadrature rule. Singular integrals of the constant and linear BEMs are done analytically, and for the quadratic use the straight line approximation of the curve near the singularity, described in [1].

The examples used for comparison here are solving for $\phi = \ln(x^2+y^2)+x$ on various regions by the constant, linear and quadratic BEMs, and the BICEM. Figures 2-5 show the results on the rectangle between $(1, 0)$ and $(5, 2)$, with ϕ given on vertical, and $\partial\phi/\partial n$ on horizontal sides. The cubic splines for the boundary values for the BICEM have 41 points per side. It was found in this work that the convergence of the BICEM to the exact solution was more heavily dependent on the boundary values' accuracy than was expected, and so a form of 'overkill' was used here. It should be noted that this did not add more than a few percent to the overall time required for the BICEM. Figures 6-9 show results on the circle, centre $(3, 1)$, radius 2, with ϕ known on the upper semicircle, $\partial\phi/\partial n$ on the lower, using 128 points for the splines. Figures 2 and 6 compare maximum error in ϕ and $\partial\phi/\partial n$ on the surface to number of unknowns, Figures 3 and 7 compare maximum error to time taken (seconds on a 12 MHz 286 PC), Figures 4 and 8 compare error at some internal points to number of unknowns, and Figures 5 and

9 compare error at some internal points to time taken. Figure 5 uses the internal points $(1.125, 1)$, $(3, 1)$, and Figure 9 uses $(3, 1)$, $(3, -0.9)$.

We can immediately see from Figures 2 and 6 that the convergence rate of the BICEM is superior to that of the traditional BEMs. We also see the comparison of the various element methods, and conclude that higher order elements are superior, for the same number of unknowns. In fact, we can consider the BICEM to be increasing the polynomial approximation over large, but because of the splines still accurately defined, elements. The hiccup for the BICEM in Figure 6 is worrying, but at the moment we feel this is a result of not enough accuracy in integrations, leading to 'noise' effects. Note that this case has only two expansions, to the four of the rectangular case. Further investigation in this area is being undertaken. It should also be noted that the maximum error in normal derivative in Figure 6 for the element methods was in fact at a position where normal derivative was defined, showing the importance of more accurate discretisations for the surface.

Figures 3 and 7 paint a slightly bleaker picture; when time taken as opposed to number of unknowns is the criteria, we see the effect of the more complicated BICEM integrals. However, after the confused middle region of the graphs, the BICEM still is the most accurate method for lower error requirements. However, we must realise that the routine DQAGS used by the BICEM is reliable, but slow. Further work is being concentrated in this area. The results of Figures 4, 8 and 5, 9 show a similar result for the solution at internal points; the BICEM converges faster than the BEMs, but suffers from the time required.

Relative merits of BICEM vs BEM

We have seen from above that the BICEM has a much better convergence rate than the standard BEM. While the BICEM is less accurate for a small number of unknowns, its convergence rate is such that the method is still superior, even though the BICEM is slower for the same number of unknowns, compared to the various BEMs. Improving the integration routines for the BICEM should lead to an improvement in its speed, and has a high priority. It can be shown that the better convergence of expansion methods over element methods still holds even for approximating functions with large variation (eg $(1/(0.1 + x)$ on $[0, 1]$) or mild endpoint singularities (eg $\sqrt{x}$ on $[0, 1]$). While expansion methods are generally much poorer with discontinuities within the region of application, this possibility has been eliminated by the choice of splitting the surface at corners or changes of boundary conditions.

While convergence itself is a good enough resason for using the BICEM, there are a number of other advantages to it. This is not a problem spe-

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Figure 2: Number of unknowns vs. maximum error in solutions for the rectangular case

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Figure 3: Time taken vs. maximum error in solutions for the rectangular case

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Figure 4: Number of unknowns vs. error in solutions at the internal points $(3, 1)$, $(1.125, 1)$ for the rectangular case

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Figure 5: Time taken vs. error in solutions at the internal points $(3, 1)$, $(1.125, 1)$ for the rectangular case

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Figure 6: Number of unknowns vs. maximum error in solutions for the circular case

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Figure 7: Time taken vs. maximum error in solutions for the circular case

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Figure 8: Number of unknowns vs. error in solutions at the internal points $(3, 1)$, $(3, -0.9)$ for the circular case

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Figure 9: Time taken vs. error in solutions at the internal points $(3, 1)$, $(3, -0.9)$ for the circular case

cific method, and so like the BEM can be used as a 'black box' to solve general potential problems. However, with the BICEM the surface and boundary value descriptions are separate from the unknowns. This gives the advantage that changing the size of the problem means merely changing the handful of numbers in the input file which are the number of unknowns on each surface. Despite the care required in setting up enough boundary points for the boundary values to be very accurate, compare this to the not insignificant problem of, for example, doubling the number of unknowns in a standard BEM formulation. We also avoid the problems corners bring in linear and higher order BEMs, by never actually placing a collocation point at a corner. Finally, as with all expansion methods, examination of the solution produced by the BICEM can give a good idea of the accuracy of the solution; the closer the last terms of an expansion are to zero (taking into account the error in the integrations), the better the approximation is to the solution. A discussion of this can be found in [6].

CONCLUSION

We have developed a method of solving the Boundary Integral Equations by an expansion method. This has been shown to be superior to the standard elements approach, without loss of generality. The various other advantages of the BICEM over the BEM have been mentioned.

Apart from improving integration routines, further work on the BICEM can be taken in various directions. Using the qualities of expansion methods, some sort of error estimation for general problems should be possible. Since changing the problem size is so simple, as compared to the BEM, some form of extrapolation leading to more accurate solutions can be investigated. This current work has been for potential problems, but changing Green's functions to allow for the solution of other problems, for example elasticity problems, should be fairly straightforward. Finally, and ultimately of most interest, we can move to 3D problems, using a double expansion for the solution over surfaces.

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