Predicting Improper Fractional Base Integer Characteristics

Billy Dorminy

Sola Fide School

Mathematics

Senior Division

Abstract

Combinatorics and algebra have been used to find equations for the smallest integer with a certain length in an integral base. However, improper fractional bases have not been explored in much depth since their discovery in the 1930s. In this study, I discovered an original formula for the smallest integer with a specific digit length in an improper fractional base.

I wrote an original computer program to convert integers from base 10 to any improper fractional base. I used this program to find 100 combinations of length, improper fractional base, and the smallest integer with that length in that fractional base. I used graphing, combinatorics, and difference equations to attempt to find a method to predict the smallest integer with a specific length in an improper fractional base.

I then used number theory to evaluate the divisibility requirements of the numbers, and discovered a recursive formula for the smallest integer with a specific length in a given improper fractional base. I used this formula to find an equation for the number of integers in an improper fractional base with a certain length. The formula may also be useful in encryption with improper fractional bases.

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1 Introduction

Since the beginning of civilization, systems of numeration have been important to mankind. People still use basic one-to-one correspondence with our fingers to find base 10. As society increased in complexity, so did our systems of numeration. Today we have a wide variety of counting methods which are no longer directly dependent upon our physical features.

1.1 Common Bases

Integral (whole number) bases are the most common. Historically, bases 10, 20, 60, and 12 have been used most widely; other bases occasionally used are 2, 8, 16, and 3.

1.1.1 Historical Bases

1.1.1.1 Bases 10, 20, and 60

We usually have 10 fingers and 10 toes, therefore it is convenient to use base 10. The Phoenicians and the Mayans used base 20, presumably counting both fingers and toes.[4] The Sumerians used base 60 for numbers over 20,[15] which is the origin of our minute and degree measures. 60 is a useful base because it has a large number of factors. Of the fractions in the form $\frac{1}{x}$ with $0 < x \leq 60$, in base 60, 12 have 1-digit decimal representations. This makes it much easier to convert between hours and minutes. In base 60, 1 hour 5 minutes is equal to 1.05; hours instead of $1.\overline{3}$ hours in base 10 (Base 60 numbers are written with a comma or semicolon separating the base-60 digits).

1.1.1.2 Base 12

Some early civilizations used the 3 joints on their 4 fingers of one hand to arrive at base 12.[9] Similar to 60, 12 has many factors. Of the first 12 natural numbers' reciprocals, only 6 do not have finite decimal representations. Base 12 has influenced us widely: for instance, 12 items is a dozen and 12 dozen is a gross.

1.1.2 Computational Bases

A computer's storage is similar to a great number of on-off switches. These are also known as 2-state devices since they have 2 possible states, corresponding to 1 and 0 in base 2. It is impossible to store more than a 1 or a 0 on a single 2-state device, therefore computers convert to base 2 to store information.[5]

Base 8 is used when a user-defined number has a power of 8 (but not of 16) as an upper bound.[3] When base 8 is used for such an application, only a third as many digits are needed to represent the number as there are in base 2. As a result, base 8 numbers are easier to read compared to base 2. Base 16 is used for applications when the upper bound is a power of 16; since four base 2 digits can be represented in one base 16 digit, the number is more compact compared to base 2 or 8, even further improving readability.[14]

1.2 Unusual Bases

1.2.1 Bases 3 and -2

Bases 3 and negative 2 were used in some early computers.[2] Base 3 is regarded by Hayes as the most efficient integral base, since it has the theoretically smallest ratio of the average length of the numbers to the number of symbols used as digits of integral bases.[8]

Base -2 was attempted in some computers. Simple multiplication and division were more difficult than in base 2, so it was abandoned. At that time, computers required an extra bit to represent the sign of a number. Although base -2 had the advantage that no bit was needed to represent the negative sign, the space savings were negligible and the problems with arithmetic make it less efficient.[13] Present computers can represent the sign of a number without using an extra bit, negating any space-saving advantage of base -2.

3-state devices are not base 3 in the traditional sense of the word. 3-state devices use what is called "balanced ternary": instead of having values 0, 1, or 2, a 3-state device can have value of either -1, 0, or 1. Thus computers using base 3 take only 1 evaluation to determine whether a number is larger, smaller, or equal to another number, yielding values -1, 0, or 1. Although base 3 is more efficient in theory, it has been difficult and expensive to implement [6]; 3-state devices have been too inefficient to compete with base 2.[10] Currently, base 2 is more practical since 2-state devices have been optimized since base 2 was first used for computing, and thus 2-state devices are very reliable and cheap. However, recent advances in 3-state hard drives have made base 3 computing much closer than before.[7]

1.2.2 Irrational Bases

An irrational number has an infinitely long, non-repeating decimal portion; it cannot be represented as a ratio of two integers.[11, pp.102-104] For instance, π (pi, the ratio of a circumference of a circle to its diameter) is an irrational number since it cannot be represented as a ratio of two integers.[16] Bergman investigated irrational bases in 1957; he discovered that numbers do not necessarily have a unique representation in an irrational base.[1]His work has been important in developing faster algorithms in computer science.[10] Though not common, irrational bases are used occasionally; base ϕ (the golden ratio, phi, $\frac{1+\sqrt{5}}{2}$) is the most common irrational base.[17]

1.2.3 Transcendental Bases

By definition, a transcendental number cannot be a solution of a polynomial with integral coefficients.[18] All transcendentals are irrational numbers, but not all irrational numbers are transcendental. π and e (the base of natural logarithms) are transcendental, but ϕ is only irrational. [16] ϕ is a root of $x^2 - x - 1 = 0$ while π and e cannot be roots of a polynomial with integral coefficients.[12, p.114]

Since transcendental bases are impractical, they are not often used. Every integer greater than the transcendental base has infinitely many digits in its representation and only multiples of the transcendental base have finite decimal representations. However, it has been shown that base e is theoretically the most efficient base out of every possible base.[19]

1.2.4 Improper Fractional Bases

An improper fractional base is rational, expressible as a ratio of two integers, and greater than 1. Improper fractional bases, such as $\frac{3}{2}$, were discovered by A. J. Kempner in 1936.[9] Besides the method of conversion to and from fractional bases, little thought had been given to these bases.

1.3 Smallest Integer Patterns

One of the main differences between integral bases and improper fractional bases is in the smallest number with a specific length in a given base. In an integral base, these numbers follow a exponential progression for digit lengths greater than 1. In base 10, the smallest integer with each number of digits is multiplied by a factor of 10 between n digits and $n+1$ digits for $n \geq 2$. The smallest 1-digit positive integers is 1, 2-digit integer is 10, of 3, 100, and so on.

In an improper fractional base, the numbers are extremely variable even from one base to another similar non-reducable base. For fractional base integers there is no obvious pattern. In base $\frac{3}{2}$, the smallest 1-digit integer is 0, the smallest 2-digit integer is 3, 3-digit integer, 6, 4-digit, 9, 5-digit, 15, and the smallest 6-digit integer is 24.

Combinatoric techniques yield an equation which predicts the smallest integer in the integral base b. In an integral base, any digit can be in any position in a number, except 0 cannot be the first digit. Thus in base b, there are $b-1$ 1-digit numbers (not including 0), $(b-1)b$ 2-digit numbers, $(b-1)b^2$ 3-digit numbers, etc. Let the number of integers with length k in base b be $g(l)$. Then the smallest integer with length l in base b is $1 + g(1) + g(2) + g(3) + \cdots + g(l-1)$, since this smallest integer will come immediately after all the integers with length less than l. This may be expressed as:

$$1 + \sum_{j=1}^{l-1} g(j)$$

Now there are $b-1$ choices of digit for the first digit of a number in base b and b choices of digit for each subsequent digit in the number, thus $g(l) = (b-1)b^{l-1}$. Thus the smallest number with l digits in base b is equal to

$$\sum_{j=1}^{l-1} b(b-1)^{j-1} = (b-1)\sum_{j=1}^{l-1} b^{j-1} = (b-1)\frac{b^{l-1}-1}{b-1} = \boxed{b^{l-1}}$$

Distressingly, this equation does not work for fractional bases.

1.4 Question

Is there an equation for the smallest integer with a given number of digits (length) in any fractional base? I sought to find such an equation which uses the fractional base and the desired length.

2 Using and Converting Bases

A base, also known as a radix, usually indicates how many unique symbols may be used to form a number. In base 10, a digit in a number could be 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9, for a total of 10 unique symbols. In base 3, a digit could be either 0, 1, or 2.[19]

Fractional bases, expressible in the form $\frac{p}{q}$, are somewhat different. Instead of using digits up to $\frac{p}{q} - 1$, in which there would be a fractional number of digits, we use p unique digits: $0, 1, 2, 3, \cdots, p-2, p-1$.[21] For instance, in base $\frac{3}{2}$, we can use the digits 0, 1, and 2. We generally denote a base b number n with n_b, read "n base b".

2.1 Conversion Using Powers of a Base

There are several methods of converting an integer n to and from base 10. The most common involves powers of the new base. To convert from base 10 to base b using this method, find the largest power of b which is smaller than n and let the exponent be i. The leftmost digit is the largest number which, when multiplied by b^i, is smaller than n. We then subtract this product from n and make this the new n and subtract 1 from i and let this be the new i.[19]

Then, until $i = 0$, we repeatedly find the largest number whose product with b^i is smaller than n, and let this number be the next digit to the right of those which we have found. Next, subtract the product of that number and b^i from n, and subtract 1 from i and make this i. When $i = 0$, we let the remaining number be the last (rightmost) digit. In base 10, for instance, the number 37 is equivalent to $(3 \cdot 10^1) + (7 \cdot 10^0)$. (Generally $x^0 = 1$, with the exception of $x = 0$.)

To convert n back to base 10, we count how many digits n has. We subtract 1 from this number of digits and let it be i. We then add the product of the first digit and b^i, the second digit and b^{i-1}, the third and b^{i-2}, and so on. This number is the base 10 representation of n_b. 1101_3 (1101, base 3) is equal to $(1 \cdot 3^3) + (1 \cdot 3^2) + (0 \cdot 3^1) + (1 \cdot 3^0)$, which is 37 in base 10.

The above method, while most common, only works for integral bases; when we attempt to use this method for an improper fractional base, the number ceases to be integral. For example, to convert 3 (base 10) to base $\frac{3}{2}$

by the above method, we find that $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$. The largest number by which this can be multiplied to get a number smaller than 3 is 1, so the leftmost digit is 1. The remainder is $\frac{3}{4}$, and the next smaller power is $\frac{3}{2}$, so the next digit is 0. The next smaller power is 1, so the third digit is also 0. But now to the right of the decimal point we must represent $\frac{3}{4}$: we cannot represent this number without a decimal point using this conversion technique. It is logical that an integer should be represented without any digits to the right of the decimal point in any rational base, though, so this method cannot be used for improper fractional bases.

2.2 Converting Fractional Bases

Another method for conversion, which fulfills the requirements to produce integers in fractional bases, follows. Designate the desired base $\frac{p}{q}$ number $\ldots d_5 d_4 d_3 d_2 d_1 d_0$ where d_k represent the digits. We then let the original number in base 10 be d_0. After this, while any of the d_k are greater than or equal to p, we subtract p from d_k and add q to d_{k+1}.[21]

For instance, to convert 19 from base 10 to base $\frac{3}{2}$, $d_0 = 19$. Instead of using d_k, we must simply separate the digits in the base b number somehow. We can use a line to separate the digits, as:

$$\mid \cdots \mid d_5 \mid d_4 \mid d_3 \mid d_2 \mid d_1 \mid d_0 \mid$$

Initially we have:

$$\mid \; \mid \; \mid \; \mid \; \mid 19 \mid$$

We subtract 6 3's from 19 and add $6 \cdot 2$ to the digit immediately to its left:

$$\mid \; \mid \; \mid \; \mid 12 \mid 1 \mid$$

Since $12 > 3$ we again subtract 3's until the digit is smaller than 3. We subtract 4 3's from 12 and add 8 to the digit immediately to the left:

$$\mid \; \mid \; \mid 8 \mid 0 \mid 1 \mid$$

Again, we subtract 6 from 8 and add 4:

	4	2	0	1

and again. $4 > 3$, so we subtract a 3 and add a 2, for a final result of

2	1	2	0	1

or $21201_{\frac{3}{2}}$ (base $\frac{3}{2}$).We generally denote a base b number n with n_b, read "n base b".

We can use this process in reverse to convert to base 10: we separate the digits, just as before. Then, starting at the left, we multiply each digit by the base b and add this to the digit immediately right. For instance, to convert 21201 from base $\frac{3}{2}$ to base 10 we separate the digits:

2	1	2	0	1

Multiply the first by $\frac{3}{2}$ and add to the second:

	4	2	0	1

Multiply the (new) first digit by $\frac{3}{2}$ and add to the (new) second digit:

		8	0	1

Multiply yet again:

			12	1

Finally multiply the first digit by $\frac{3}{2}$ and add to the second digit to get $21201_{\frac{3}{2}} = 19_{10}$.

3 Materials and Procedures

3.1 Conversion Program

I created an original computer program and converted large numbers of base 10 integers to various fractional bases. I used tools from combinatorics, number theory, and polynomial analysis to find an equation appropriate to the data.

I wrote this program in Python:

```
def DecimalToFractionalBaseConversion(DecimalNumber,
                    FractionalBaseNumerator,FractionalBaseDenominator):
ArrayOfDigits=[DecimalNumber]
while ArrayOfDigits[0]>=FractionalBaseNumerator:
ArrayOfDigits=[ArrayOfDigits[0]]+ArrayOfDigits
ArrayOfDigits[1]=ArrayOfDigits[1] % FractionalBaseNumerator
ArrayOfDigits[0]=ArrayOfDigits[0]-ArrayOfDigits[1]
ArrayOfDigits[0]=ArrayOfDigits[0]*FractionalBaseDenominator
ArrayOfDigits[0]=ArrayOfDigits[0]/FractionalBaseNumerator
return ArrayOfDigits
```

My program above takes as input the base 10 integer, the improper fractional bases' numerator, and the improper fractional bases' denominator. The program then creates ArrayOfDigits, which will be used as an array. Initially, ArrayOfDigits contains one element. Instead of making its boxes at the beginning of the conversion process, as we do when converting to an improper fractional base, ArrayOfDigits adds extra boxes only when needed. As in our method of conversion, initially the first box of ArrayOfDigits contains the base 10 integer.

Then the program checks to see if the first member of ArrayOfDigits, the leftmost box, is bigger than FractionalBaseNumerator, the numerator of the fractional base. If it is, another member or "box" is added to the beginning of ArrayOfDigits.

The program next copies the former first member of ArrayOfDigits to the new first member of ArrayOfDigits. The second member of ArrayOfDigits is taken modulus FractionalBaseNumerator, which is simply the remainder when that member is divided by the numerator of the fractional base. This, the second member, is subtracted from the first member, so now the first member is divisible by FractionalBaseNumerator, and the first member is multiplied by the inverse of the base. This is the same as subtracting the numerator and adding the denominator to the next digit for those digits. The program checks again to see if the first member is bigger than the numerator, and repeats if this is true.

When all of the members are smaller than the numerator of the fractional base, the program returns ArrayOfDigits - the contents of our boxes when we are done converting.

3.2 Bases Used

I used this program to convert the first 5000 positive integers to bases $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{7}{3}$, $\frac{7}{4}$, $\frac{7}{5}$, and $\frac{8}{6}$. In each of these bases, I found the smallest integer in that base with various lengths (1 digit, 2 digits, 3 digits, etc.). A fractional base, a length, and the smallest integer in that base with exactly that length formed 100 data sets.

- I chose base $\frac{3}{2}$ since it has the smallest possible irreducible numerator and denominator for an improper number, thus it is easiest to examine of improper fractional bases.

- I chose bases $\frac{4}{3}$ and $\frac{5}{4}$ since the denominators are only 1 smaller than the numerators. I could easily test an equation for base $\frac{3}{2}$ with a similar base to find the proper generalization for any base with the numerator 1 larger than the denominator.

- I chose bases $\frac{7}{3}$, $\frac{7}{4}$, and $\frac{7}{5}$ since the denominators were more than 1 smaller than the numerator. It would be possible to generate a equation that would work for the bases with the numerator 1 greater than the denominators, but it would be very hard to find an equation that works for both those 3 bases with numerator 7 and the 3 bases with the numerator 1 greater than the denominator.

- Finally, I chose base $\frac{8}{6}$ since it is a reducable improper fraction. I wanted to test any possible equation on at least one reducable fractional base. If a base were not in lowest terms, an equation dependent on the base being in lowest terms would fail for a reducable improper fractional base. I wanted to insure my equation was not dependent on this condition.

3.3 Methodology

I initially used difference equations to search for the next member of the sequence of smallest integers with various lengths and fractional bases. However, in all tests the value predicted by the difference equations differed from the correct value by a significant amount. Given $d + 1$ values of any polynomial with degree d, all other values of the polynomial may be predicted. Since the difference equations failed with up to 10 values, evidently the equation is not a polynomial or has degree more than 9.

If the graphs of the smallest integers with n digits in the above bases were similar, I would expect the equation to be a polynomial. Although all graphs were somewhat exponentially curved, the similarities were inconsequential. The negative results from both graphing and difference analysis indicate that the desired equation is not a polynomial. The piecewise nature of the graphs suggested floor or ceiling functions were at work.

Combinatoric techniques have been used in the past to find the equation for the smallest integer with a specific length in an integral base. I hoped similar methods could be used to find the desired formula. I tried to find the formula by using base p and then, one digit at a time, converting to base $\frac{p}{q}$ in the hope that a pattern would emerge. Sadly, I found no pattern between the conversions to $\frac{p}{q}$.

Eventually, I employed number theory to discover the method of forming the smallest number with n digits, by examination of divisibility requirements. I then tested this recursive formula with all data created by the program.

4 Results

I discovered some divisibility properties of improper fractional base integers. With this knowledge, I discovered an original formula for the smallest integer with a specific length in an improper fractional base, which showed my hypothesis to be true. I also discovered an original equation for the number of integers with a specific length in an improper fractional base, and a new method of encryption.

Theorem. *For any improper fractional base $\frac{p}{q}$, the smallest integer with n digits in its representation in base $\frac{p}{q}$ is*

$$\frac{p\left(f^{n-2}(q)\right)}{q}$$

where $f(x) = q\left\lceil \frac{p \cdot x}{q^2} \right\rceil$ *and* $\lceil x \rceil$ *is the smallest integer larger than or equal to* x.

(To find $f^a(b)$, substitute b into $f(x)$. Substitute this result into $f(x)$ again, and substitute that result into $f(x)$ yet again, until we have applied $f(x)$ a times to b.)

Lemma. *Let an integer in base $\frac{p}{q}$ be represented as $d_0d_1d_2 \cdots d_j$, where d_k is the $(k-1)$th digit to the right of the rightmost digit. Then all of the integers d_0, d_0d_1, $d_0d_1d_2$, $\cdots$, $d_0d_1d_2 \cdots d_{j-1}$ are divisible by q.*

Proof. Each of the integers d_0, d_0d_1, $d_0d_1d_2$,$\cdots$, $d_0d_1d_2 \cdots d_{j-1}$ may be represented by $d_0 \cdots d_m$, where m is a positive integer. According to our method of conversion from base 10 to a fractional base, each of these numbers $d_0 \cdots d_m$ is created by subtracting p from d_{m+1} and adding q to d_m repeatedly. Thus $d_0 \cdots d_m$ in base $\frac{p}{q}$ can be represented as qn, where n is the number of subtractions of p from d_{m+1}; since n is integral, $q \mid qn$, as desired. □

Proof of Theorem. The smallest number with exactly n digits will have a first digit of q, since d_0 must be divisible by q. Every subsequent digit d_m up to but not including the last digit on the right d_n will be the smallest such that the number $d_0d_1d_2 \cdots d_m$ is divisible by q.

If we have already found the smallest possible number $d_0d_1d_2\cdots d_{m-1}$ and we wish to find the smallest possible integer $d_0d_1d_2\cdots d_m$, we append a 0 on the end of this number and find the smallest possible d_m so that $q \mid d_0d_1d_2\cdots d_{m-1}0 + d_m$. This is the smallest possible digit d_m such that $d_0d_1d_2\cdots d_m$ is divisible by q.

When we append a 0 on the end of the integer $d_0d_1d_2\cdots d_{m-1}$, we are simply shifting the digits of the integer 1 place to the left, which is equivalent to multiplying the number by our base $\frac{p}{q}$. Thus d_m is the smallest possible integer such that
$q \mid (d_0d_1d_2\cdots d_{m-1})\frac{p}{q}+d_m$. This process is equivalent to dividing $(d_0d_1d_2\cdots d_{m-1})\,\frac{p}{q}$ by q, rounding up, and multiplying the result by q, giving the number $d_0d_1d_2\cdots d_m$. Mathematically,

$$d_0d_1d_2\cdots d_m = q\left\lceil \frac{(d_0d_1d_2\cdots d_{m-1})\,p}{q^2}\right\rceil$$

This can be expressed as a function which takes as input $d_0d_1d_2\cdots d_{m-1}$ and returns $d_0d_1d_2\cdots d_m$:

$$f(x) = q\left\lceil \frac{xp}{q^2}\right\rceil$$

If we take the starting digit $d_0 = q$ of the smallest number, when we apply $f(x)$ to q, we find the first 2 digits of the number; if we apply it to this result we get the first 3 digits, and so forth. However, the whole number $d_0d_1d_2\cdots d_n$ does not have to be divisible by q, so we cannot apply $f(x)$ to $d_0d_1d_2\cdots d_{n-1}$ to find $d_0d_1d_2\cdots d_n$. Instead, the smallest number with n digits ends in 0, so to find $d_0d_1d_2\cdots d_n$ we multiply $d_0d_1d_2\cdots d_{n-1}$ by $\frac{p}{q}$, our base.

The application of a function f to an input i z times is denoted by $f^z(i)$. For instance, $f(f(f(5))) = f^3(5)$. When we apply the function above $n-2$ times, we find the integer $d_0d_1d_2\cdots d_{n-1}$. To find the full number $d_0d_1d_2\cdots d_n$, we multiply $d_0d_1d_2\cdots d_{n-1}$ by $\frac{p}{q}$. Thus the smallest number with n digits is:

$$\frac{p\cdot f^{n-2}(q)}{q} \text{ where } f(x) = q\left\lceil \frac{xp}{q^2}\right\rceil$$

□

This novel equation was tested with all 100 sets of data and will work for any rational base and length.

4.1 Extension

This formula may also be used to find the number of integers with a specific length. To find the number of integers with a specific length in an improper fractional base, we can take the difference of the smallest integer with that length and the smallest integer with one more digit. Thus the number of integers in an improper fractional base with n digits in the improper fractional base $\frac{p}{q}$ is equal to

$$\frac{p\left(f^{n-1}(q) - f^{n-2}(q)\right)}{q}$$

where $f(x) = q\left\lceil \frac{px}{q^2} \right\rceil$.

5 Encryption Applications

Fractional bases may encrypt data by converting the message to a number, then to a fractional base, and then finally breaking it up and converting it back to a string of letters. My equation is useful to fractional base encryption when a long message must be split into several parts.

5.1 Fractional Base Encryption of Short Messages

Let us encrypt "Word Power": A space may be represented as 00, 'a' can be 01, 'b', 02, etc. up to 'z' being 26, with lowercase and uppercase letters identical. The numerical representation of this message is thus "23151804001615230518".

A fractional base with no more than 27 digits is necessary to encrypt this message, since there are 27 possible characters; if we used a base whose numerator is less than 27, the encrypted message would be longer and would have fewer unique characters. If we used a base whose numerator was more than 27, it is possible that a digit more than 26 would appear in the number in that base; we only have 27 characters, so that digit could not be converted to a character. The base $\frac{27}{2}$ is fairly quick to evaluate, and we find that our number is:

| 20 | 21 | 09 | 23 | 12 | 24 | 06 | 25 | 01 | 04 | 25 | 25 | 07 | 10 | 13 | 18 |

in base $\frac{27}{2}$. Now converting the digits into the corresponding letters, our message ends up as 'tuiwlxfyadyygjmr'.

To decrypt this encrypted message, convert the encrypted message 'tuiwlxfyadyygjmr' back to the base $\frac{27}{2}$ number

| 20 | 21 | 09 | 23 | 12 | 24 | 06 | 25 | 01 | 04 | 25 | 25 | 07 | 10 | 13 | 18 |

Then convert this number to base 10 and convert back to letters by our original method.

5.2 Encryption of Long Messages

If a long message is to be encrypted, it must be divided into several sections to encrypt separately, then combined into one string. After splitting the message into sections, it is necessary to find a base such that all of the encrypted sections have the same length. If the sections differ in length, then the message is undecipherable after the sections are combined. When the sections have equal length, the recipient deciphers the message using the number of sections or their length and the base.

It is necessary to use a base such that all of the numerical representations of the sections have the same length in that base. Few bases are capable of fulfilling this requirement for a given set of sections. Trial and error can be used to find a base such that all the encoded sections have the same length. In my research, I discovered an original equation which predicts the smallest integer in a fractional base. This equation may be used to determine if all the encrypted sections in that base are of equal length.

For instance, we have the message "ISEF Participants in Indiana" to encrypt using this method. We convert this message to the numerical representation

$$0919050600160118200903091601142019000914000914040901 1401$$

Evidently this would take a prohibitive amount of time to encode, so we split it into sections. We attempt the base $\frac{729}{2}$: 729 is 27^2, thus every digit in base 729 can be represented by a combination of 2 characters.

My original equation for the smallest integer with a given length in an improper fractional base can be used to much advantage. We use it to find that every base $\frac{729}{2}$ integer with 6 digits is at least 12,885,896,162,367 in base 10, and every base $\frac{729}{2}$ integer with 7 digits is at least 4,696,909,151,183,136 in base 10. Thus if we split the message into 16-digit sections, each of the sections will be equally long when encrypted:

$$0919050600160118$$

$$2009030916011420$$

$$1900091400091404$$

$$0901140100000000$$

We may add some spaces - pairs of zeros - to the end of the message, to make all the unencrypted sections the same length, so it is easier to decrypt.

Leading zeros on the sections are dropped, so if the recipient knows all the sections are the same length, the missing zeros can be replaced.

Now we convert each of these numbers to base $\frac{729}{2}$:

$$142; 305; 512; 704; 016; 200$$

$$312; 090; 075; 601; 080; 632$$

$$294; 479; 235; 068; 580; 645$$

$$140; 020; 275; 320; 194; 007$$

We then convert each of these digits to their 2-digit base-27 representation:

$$05, 07; 11, 08; 18, 26; 26, 02; 00, 16; 07, 11$$

$$11, 15; 03, 09; 02, 21; 22, 07; 02, 26; 23, 11$$

$$10, 24; 17, 20; 08, 19; 02, 14; 21, 13; 23, 24$$

$$05, 05; 00, 20; 10, 05; 11, 23; 07, 05; 00, 07$$

We finally convert each of these digits into the corresponding character and join the message. As before, '00' is equal to a space, '01', 'a', etc.:

'egkhrzzb pgkkocibuvgbzwkjxqthsbnumwxee tjekwge g'

To decrypt the message, the recipient must know that $\frac{729}{2}$ was used as the base and that the number was split into 4 sections to encrypt. Even if someone knew the base was $\frac{729}{2}$, the message is 48 characters long. There are 8 lengths possible to split up the message, but only one of these will give the correct result.

5.3 Security Issues

Breaking a message encrypted in this manner requires testing possible bases and section lengths until the correct combination is found. The denominator must divide the first digit of the fractionally based integer evenly, thus if the first digit is known, the possible denominators can be limited to divisors of that digit. The numerator of the fractional base must be at least as large as the number of unique digits in the numerical representation of the encrypted message, thus if the numerical representation of the encrypted message in the improper fractional base is known, the numerator of the fractional base is limited to a few values.

Using a base with a numerator greater than the number of characters available and using combinations of characters to represent the digits can make it much more difficult to crack the message. As groups of characters represent one digit, the first digit is much more difficult to find, thus finding the correct denominator is also more difficult. For the same reason, it is harder to find the number of unique digits, thus making it also more difficult to narrow the possible numerators.

6 Conclusions

I created an original formula which predicted the smallest integer with a given length in a fractional base from the base and length, in accordance with the hypothesis. The formula was tested for 100 sets of data and is conjectured to work for all fractional bases.

From this formula I created another original equation yielding the number of integers with a specific length in an improper fractional base.I also discovered a new method of encryption using fractional bases, in which my equation is useful.

Since fractional bases were discovered in 1936, little had been investigated beyond the method of integer conversion. The uses of fractional bases are open to exploration and will receive continued attention in coming years.

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9 Official Abstract and Certification

10 Checklist for Adult Sponsor / Safety Assessment Form (1)

11 Research Plan (1A)

12 Research Plan Attachment

13 Approval Form (1B)