OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth. 2003; **0**:1–14 *Prepared using* ocaauth.cls *[Version: 2002/11/11 v1.00]*

Computations for Bang–Bang Constrained Optimal Control Using a Mathematical Programming Formulation

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SUMMARY

An algorithm is proposed to solve the problem of bang–bang constrained optimal control of nonlinear systems with free terminal time. The initial and terminal states are prescribed. The problem is reduced to minimising a Lagrangian subject to equality constraints defined by the terminal state. A solution is obtained by solving a system of nonlinear equations. Since the terminal time is free, time-optimal control is given a special emphasis. Second-order sufficient conditions of optimality are also stated. The algorithm is demonstrated by a detailed study of the switching structure for stabilizing the F–8 aircraft in minimum time, and other examples. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: optimal control, bang-bang control, Lagrangians, second-order sufficiency, F–8 aircraft

1. Introduction

Bang-bang control, where the input switches between upper and lower bounds, is the optimal strategy for solving a wide variety of control problems, where the control of a dynamical system has lower and upper bounds, and the system model is nonsingular and linear in the input. Bang–bang control is often the appropriate choice because of the nature of the 'actuator' of the physical system, such as the on–off positions of a thermostat. In more general problems, where it is necessary to switch from one mode of the system to another, this switching may be modelled by a bang–bang type control. Bang–bang type controls arise in well-known application areas such as robotics, rocket flight, and cranes, and also in applied physics [1], game theory [2], chemistry [3], biological [4, 5] and socio-economic systems [6]. Optimal bang–bang control has been considered in a number of algorithms reported in the literature, citations of which can be found in Kaya and Noakes [7] and Scrivener and Thompson [8]. Examples include the *switching-time-variation-method* due to Mohler [9, 10], the *switch time optimization* algorithm by Meier and Bryson [11], the *control parametrization enhancing technique* by Lee

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et al. [12], the *enhanced transcription scheme* by Hu *et al.* [13], a smoothing technique by Bertrand and Epenoy [14], and the *time-optimal switchings* (TOS) algorithm due to Kaya and Noakes [15].

In this paper we propose a numerical technique to solve a class of bang–bang constrained optimal control problems, extending that given in Simakov *et al.* [16], where the focus was on time-optimal control and details were omitted. Here, we provide these details and further consider the treatment of more general performance indices than just minimizing terminal time. We first reduce the optimal control problem to the problem of minimizing a Lagrangian subject to an equality constraint defined by the terminal state in the arc-times space. In the minimization of the Lagrangian, gradient calculations are carried out in a method similar to that used for the switching time calculations (STC) method in Kaya and Noakes [7]. Lucas and Kaya [17] present a version of STC where instead of minimizing the distance from the terminal point, a nonlinear system of equations is solved. The Lagrangian minimization problem also reduces to solving a nonlinear system of equations, where the numerical scheme proposed in Lucas and Kaya [17] is incorporated.

The TOS algorithm presented in Kaya and Noakes [15] needs a feasible bang–bang solution as the initial guess, which is typically obtained using STC. TOS achieves a bang–bang constrained time-optimal solution using the gradient calculations in Kaya and Noakes [7] and the projection of the gradient on the surface defined by the terminal state in the optimization space. The Lagrangian formulation given in this paper can achieve what STC and TOS can achieve together, albeit with the use of second-order variations in addition to the gradient. This additional numerical expense is necessary for finding a stationary point of the Lagrangian. On the other hand, the second-order information facilitates the process of devising a scheme and checking whether the stationary point is a minimizer. In other words, with this technique we can check, not only the necessary, but also the sufficient conditions for a minimum. Numerical experiments suggest that the proposed algorithm can handle a poor initial guess reasonably well. Furthermore, the formulation allows us to tackle general performance indices.

The paper is organized as follows. In Section 2 we give a formulation and statement of the problem in the so-called arc-times space. In Section 3 we present the Lagrangian formulation in the arc-times space and state the necessary and sufficient conditions. We also give some details of the numerical scheme we employ for solving the problem. In Section 4 we apply the computational technique to a suite of example systems, in particular the F–8 aircraft, for which a detailed study of the switching structure for different angles of attack is also presented.

2. Preliminaries and Problem Statement

Consider the bang–bang constrained optimal control problem

$$(\mathrm{P})\left\{\begin{array}{ll}\text{minimize} & \phi(\mathbf{x}(t_f),t_f)\\ \text{subject to} & \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),u(t)),\qquad u(t)\in\{-1,1\}\ ,\\ & \mathbf{x}(t_0)=\mathbf{x}_0,\quad \mathbf{x}(t_f)=\mathbf{x}_T,\end{array}\right.$$

where $t\in[t_0,t_f]$, the terminal time t_f is free, the state $\mathbf{x}:[t_0,t_f]\longrightarrow\mathbb{R}^n$ is continuous, and $\mathbf{f}:\mathbb{R}^n\times\{-1,1\}\longrightarrow\mathbb{R}^n$ is smooth in $\mathbf{x}$ except possibly at the time points where the control

$u : [t_0, t_f] \longrightarrow \{-1, 1\}$ switches between -1 and 1. The terminal cost $\phi : \mathbb{R}^n \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is also smooth in the same fashion.

The Mayer type cost used in Problem (P) can accommodate other types of cost, such as the so-called Lagrange and Bolza types. For example, if the cost to minimize is of Lagrange type, i.e. the performance criterion is minimize $\int_{t_0}^{t_f} f_0(\mathbf{x}(t), u(t))\, dt$, then the problem can easily be converted to Mayer type by defining a new state variable $x_{n+1}(t) := \int_{t_0}^{t} f_0(x(t), u(t))\, dt$. The differential equation $\dot{\mathbf{x}}_{n+1}(t) = f_0(\mathbf{x}(t), u(t))$, $\mathbf{x}_{n+1}(0) = 0$, is appended to the system dynamics. In this case, $\phi(\tilde{\mathbf{x}}(t_f), t_f) = x_{n+1}(t_f)$ with $\tilde{\mathbf{x}} = (\mathbf{x}^T, x_{n+1})^T$. Note that the terminal value of the new state, $x_{n+1}(t_f)$, will be free. A performance index which is of the more general Bolza type (a combination of Mayer and Lagrange types) can be similarly treated.

The points t_k $(k = 1, 2, \ldots)$ where u is discontinuous are called the *switching times*. Let N be the number of switchings taking place in the interval (t_0, t_f), so $t_0 < t_1 \leq \ldots \leq t_N < t_f$.

To simplify the notation, we will drop the dependence on t from the following expressions whenever appropriate, rewriting the dynamical equations in Problem (P) as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{1}$$

where $u(t) = u_k$ if $t \in [t_{k-1}, t_k)$. Bang–bang control problems can be scaled so that $u_{k+1} = -u_k$, $k = 1, \ldots, N$, with $u(t_0) = u_1 = 1$ or -1.

The control u is said to be *admissible* [18] for the prescribed initial and terminal points $\mathbf{x}_0$ and $\mathbf{x}_T$ if (1) results in a solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_T$. Such a solution is called a *feasible (*or an *admissible) trajectory.* One can construct an admissible control u with given u_k, $k = 1, \ldots, N+1$, by appropriately choosing the switching times $t_1, \ldots, t_N$ and the final time t_f.

A segment of the trajectory $\mathbf{x}(t)$ corresponding to the time interval from t_{k-1} to t_k represents a smooth arc. The dynamical system (1) can also be written as the sequence of initial value problems

$$\dot{\mathbf{x}} = \mathbf{f}_k(\mathbf{x}) \text{ for } t \in [t_{k-1}, t_{k-1} + \xi_k), \quad \mathbf{x}(t_{k-1}) = \begin{cases} \mathbf{x}_0, & \text{if } k = 1 \ ; \\ \mathbf{x}(t_{k-1}^-), & \text{if } k > 1 \ , \end{cases} \tag{2}$$

where $\mathbf{f}_k(\mathbf{x}) := \mathbf{f}(\mathbf{x}, u_k)$, $\mathbf{x}(t_{k-1}^-) = \lim_{t \to t_{k-1}^-} \mathbf{x}(t)$, and ξ_k is the time-duration of the k-th arc, or simply the k-th *arc-time*, given by

$$\xi_k = \begin{cases} t_k - t_{k-1}, & \text{if } k = 1, 2, \ldots N \ , \\ t_f - t_N, & \text{if } k = N+1 \ . \end{cases}$$

A sketch of a feasible trajectory is shown in Figure 1.

The smooth segment of a trajectory $\mathbf{x}(t)$ corresponding to the interval $[t_{k-1}, t_k]$ can be parameterized as $\mathbf{x}(t_{k-1} + \tau)$, where $\tau \in [0, \xi_k]$ $(k = 1, \ldots, N+1)$. We will use the notation $\mathbf{x}(\tau; \xi_{k-1}, \ldots, \xi_1) := \mathbf{x}(t_{k-1} + \tau)$, which explicitly shows that the behavior of $\mathbf{x}$ in the k-th arc also depends on the previous arc-times. Although $\mathbf{x}$ now appears to have two different sets of arguments (or domains) as a vector function, which creates a mathematical ambiguity, we refrain from introducing a new symbol for the sake of simplicity. Furthermore the kind of $\mathbf{x}$ we will be using should be clear from the context. Note that for the first arc the new notation

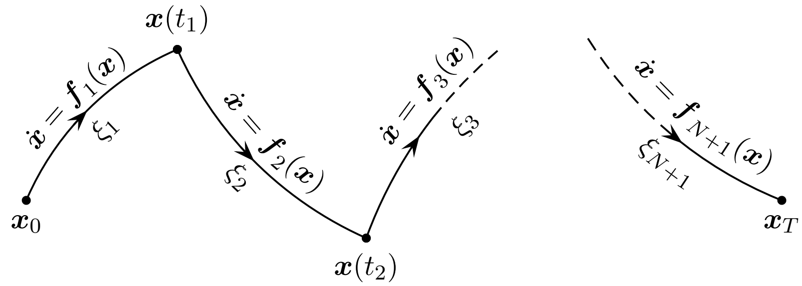

Figure 1. An admissible trajectory between $\mathbf{x}_0$ and $\mathbf{x}_T$.

is simply $\mathbf{x}(\tau)$ as there are no previous arcs. For the final point in the last arc we will write $\mathbf{x}(\boldsymbol{\xi}) := \mathbf{x}(\xi_{N+1}; \xi_N, \ldots, \xi_1)$.

Using the arc-times setting, Problem (P) can be re-written as

$$\text{(P1)} \begin{cases} \text{minimize} & \phi(\mathbf{x}(\boldsymbol{\xi}), \boldsymbol{\xi}) \\ \text{subject to} & \mathbf{x}(\boldsymbol{\xi}) = \mathbf{x}_T, \quad \xi_k \geq 0, \end{cases}$$

where $\mathbf{x}(\boldsymbol{\xi})$ is obtained by solving the sequence of initial value problems in (2) for $k = 1, \ldots, N+1$ – the $\mathbf{x}(\boldsymbol{\xi}) = \mathbf{x}_T$ are linearly independent. Assume that N and $u(0) = u_1$ are prescribed. Then Problem (P) and Problem (P1) are equivalent if $N \geq N^* + 1$, where N^* is an optimum number of switchings. In this paper a special emphasis is given to time-optimal control, for which $\phi(\mathbf{x}(\boldsymbol{\xi}), \boldsymbol{\xi}) = \xi_1 + \xi_2 + \cdots + \xi_{N+1}$.

3. Computations Using a Lagrangian Formulation

In this section we present a computational scheme for solving Problem (P1), and thus Problem (P). It will be assumed throughout that the following are specified:

(i) N, the number of switchings;
(ii) u_k, $k = 1, \ldots, N+1$, values of $u(t)$ in respective arcs; and
(iii) $\mathbf{x}_0$ and $\mathbf{x}_T$, initial and target points.

We introduce new variables $\{\alpha_i\}$ such that $\xi_i = \alpha_i^2$ for $i = 1, \ldots, N+1$. The minimization problem formulated using α_i will not involve inequality constraints as the resulting ξ_i will be always nonnegative. We will use the notation $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_{N+1})^T$, $\mathbf{x}(\boldsymbol{\alpha}) := \mathbf{x}(\boldsymbol{\xi})|_{\xi_i = \alpha_i^2; i=1,\ldots,N+1}$ and $\phi(\mathbf{x}(\boldsymbol{\alpha}), \boldsymbol{\alpha}) := \phi(\mathbf{x}(\boldsymbol{\xi}), \boldsymbol{\xi})|_{\xi_i = \alpha_i^2; i=1,\ldots,N+1}$. Then Problem (P1) becomes

$$\text{(P2)} \begin{cases} \text{minimize} & \phi(\mathbf{x}(\boldsymbol{\alpha}), \boldsymbol{\alpha}) \\ \text{subject to} & \mathbf{x}(\boldsymbol{\alpha}) = \mathbf{x}_T, \end{cases}$$

where $\mathbf{x}(\boldsymbol{\alpha})$ is obtained by solving the sequence of initial value problems in (2) for $k = 1, \ldots, N+1$. Note that for the time-optimal control problem we have $\phi(\mathbf{x}(\boldsymbol{\alpha}), \boldsymbol{\alpha}) = \boldsymbol{\alpha}^T \boldsymbol{\alpha}$.

3.1. Necessary conditions of optimality

Although the formulation given above involves ODEs, one can treat Problem (P2) as a standard optimization problem with equality constraints (see for example Chong and Żak [19]), whose Lagrangian has the form $L(\boldsymbol{\alpha};\boldsymbol{\lambda}) = \phi(\mathbf{x}(\boldsymbol{\alpha}),\boldsymbol{\alpha}) + (\mathbf{x}(\boldsymbol{\alpha}) - \mathbf{x}_T)^T\boldsymbol{\lambda}$, where $\boldsymbol{\lambda} := (\lambda_1,\ldots,\lambda_n)^T$ is the Lagrange multiplier vector. The Lagrange conditions are given by

$$\begin{cases} \nabla_{\boldsymbol{\alpha}}\phi(\mathbf{x}(\boldsymbol{\alpha}),\boldsymbol{\alpha}) + J^T(\boldsymbol{\alpha})\boldsymbol{\lambda} = \mathbf{0}, \\ \mathbf{x}(\boldsymbol{\alpha}) - \mathbf{x}_T = \mathbf{0}, \end{cases} \tag{3}$$

where $\nabla_{\boldsymbol{\alpha}}\phi = J^T(\boldsymbol{\alpha})\,(\partial\phi/\partial x)^T + (\partial\phi/\partial\alpha)^T$,

$$J(\boldsymbol{\alpha}) = \begin{pmatrix} x_{1\alpha_1} & \cdots & x_{1\alpha_{N+1}} \\ \vdots & \vdots & \vdots \\ x_{n\alpha_1} & \cdots & x_{n\alpha_{N+1}} \end{pmatrix}, \quad \text{and} \quad x_{i\alpha_j} \equiv \frac{\partial x_i}{\partial \alpha_j}\,. \tag{4}$$

The system (3) consists of $(N+1+n)$ equations in $(N+1+n)$ unknown components of $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$. A numerical solution of the Lagrange equations (3) can be obtained using some variant of Newton's method. Details of the numerical procedure are discussed in the following subsection.

Numerical details

One can combine the left-hand sides of the two vector equations in (3) into a single vector, such that $\boldsymbol{\Phi}(\boldsymbol{\alpha},\boldsymbol{\lambda}) := (\nabla_{\boldsymbol{\alpha}}\phi(\mathbf{x}(\boldsymbol{\alpha}),\boldsymbol{\alpha}) + J^T(\boldsymbol{\alpha})\boldsymbol{\lambda},\ \ \mathbf{x}(\boldsymbol{\alpha}) - \mathbf{x}_T)^T$. Now the Lagrange conditions can be rewritten in the form

$$\boldsymbol{\Phi}(\boldsymbol{\alpha},\boldsymbol{\lambda}) = \mathbf{0}\,. \tag{5}$$

If the standard Newton's method is used to solve (5), the next iterate to $(\boldsymbol{\alpha},\boldsymbol{\lambda})$ is $(\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\boldsymbol{\lambda}+\delta\boldsymbol{\lambda})$ where the update vector $(\delta\boldsymbol{\alpha},\delta\boldsymbol{\lambda})$ is found from the linear system

$$J_{\boldsymbol{\Phi}}(\boldsymbol{\alpha},\boldsymbol{\lambda})\begin{pmatrix}\delta\boldsymbol{\alpha}\\ \delta\boldsymbol{\lambda}\end{pmatrix} = -\boldsymbol{\Phi}(\boldsymbol{\alpha},\boldsymbol{\lambda}),$$

in which $J_{\boldsymbol{\Phi}}$ is the Jacobian matrix of $\boldsymbol{\Phi}$ in the form

$$J_{\boldsymbol{\Phi}} \equiv \left(\begin{array}{c|c} [L_{\alpha_i\alpha_j}] & J^T \\ \hline J & 0_{n\times n} \end{array}\right). \tag{6}$$

In (6), $[L_{\alpha_i\alpha_j}]$ is the hessian of the Lagrangian with respect to $\boldsymbol{\alpha}$. Note that

$$J = 2J_{\boldsymbol{\xi}}\,\mathrm{diag}(\alpha_1,\ldots,\alpha_{N+1}), \quad \text{where} \quad J_{\boldsymbol{\xi}} \equiv \left(\frac{\partial\mathbf{x}}{\partial\xi_1}(\boldsymbol{\xi}),\ldots,\frac{\partial\mathbf{x}}{\partial\xi_{N+1}}(\boldsymbol{\xi})\right). \tag{7}$$

The matrix $J_{\boldsymbol{\xi}}$ can be evaluated from numerical solution of the systems of ordinary differential equations derived from (2) by differentiating in respective variables. These differential equations are given in the i-th arc as [7]

$$\frac{\partial}{\partial\tau}\frac{\partial\mathbf{x}}{\partial\xi_j}(\tau;\xi_{i-1},\ldots,\xi_1)) = \frac{\partial\mathbf{f}_i}{\partial\mathbf{x}}(\mathbf{x}(\tau;\xi_{i-1},\ldots,\xi_1))\frac{\partial\mathbf{x}}{\partial\xi_j}(\tau;\xi_{i-1},\ldots,\xi_1),$$

with the initial condition

$$\frac{\partial \mathbf{x}}{\partial \xi_j}(0;\xi_{i-1},\ldots,\xi_1) = \frac{\partial \mathbf{x}}{\partial \xi_j}(\xi_{i-1};\xi_{i-2},\ldots,\xi_1)$$

for $i = 2,\ldots,N+1$, $j = 1,\ldots,N$, and $j < i$. Note that

$$\frac{\partial \mathbf{x}}{\partial \xi_{i-1}}(\xi_{i-1};\xi_{i-2},\ldots,\xi_1) = \mathbf{f}_{i-1}(\mathbf{x}(\xi_{i-1};\xi_{i-2},\ldots,\xi_1)),$$

and furthermore, at the terminal point,

$$\frac{\partial \mathbf{x}}{\partial \xi_{N+1}}(\boldsymbol{\xi}) = \mathbf{f}_{N+1}(\mathbf{x}(\boldsymbol{\xi})).$$

The above differential equations are solved simultaneously those in (2).

It is convenient at this point to adopt the following notation for computations, which is consistent with MATLAB's cell-arrays [20]. Consider a square array Y of size $(N+1)$ whose element $Y\{k,m\}$ in the k-th row and m-th column is a vector of length n. Form the k-th row of Y as

$$\begin{aligned} &\text{if } m > k, \quad Y\{k,m\} = \mathbf{0}, \\ &\text{if } m = k, \quad Y\{k,k\} = \mathbf{f}_k(\mathbf{x}(\xi_k;\xi_{k-1},\ldots,\xi_1)), \\ &\text{if } m < k, \quad Y\{k,m\} = \mathcal{R}_k(\xi_k; Y\{k-1,m\}), \end{aligned}$$

where $\mathcal{R}_k(\tau;\mathbf{y})$ denotes the solution of the initial value problem

$$\begin{cases} \dfrac{d}{d\tau}\mathcal{R}_k = \dfrac{\partial \mathbf{f}_k}{\partial \mathbf{x}}(\mathbf{x}(\tau;\xi_{k-1},\ldots,\xi_1))\mathcal{R}_k, \\ \mathcal{R}_k|_{\tau=0} = \mathbf{y}. \end{cases} \tag{8}$$

It can be verified that the m-th column of $J_{\boldsymbol{\xi}}$ is $Y\{N+1,m\}$.

Once $\mathbf{x}(\tau;\xi_{k-1},\ldots,\xi_1)$ has been found, the problem (8) represents a linear system of ODEs with varying coefficients. Simultaneous computation along the k-th arc, achieved by considering in (8) a matrix input $\mathbf{y}$ with columns $Y\{k-1, 1:k-1\}$, makes the procedure more efficient if MATLAB is used as a programming environment.

Newton's method for solving nonlinear systems of equations is rarely used in its pure form [24]. In our computer code we adopted a standard line-search modification to Newton's method [21] to improve the likelihood of convergence from an arbitrary starting point.

3.2. Sufficient conditions of optimality

If equations (3) are satisfied at a point $\boldsymbol{\alpha}_0$ (with a corresponding $\boldsymbol{\lambda}_0$), then $\boldsymbol{\alpha}_0$ is a possible minimizer. To confirm that $\boldsymbol{\alpha}_0$ is a minimizer, further investigation of the behavior of the Lagrange function at $\boldsymbol{\alpha}_0$ is required. This involves examination of the quadratic form

$$\sum_{i=1}^{N+1}\sum_{j=1}^{N+1} L_{\alpha_i\alpha_j}(\boldsymbol{\alpha}_0,\boldsymbol{\lambda}_0)\, d\alpha_i d\alpha_j, \quad \text{where} \quad L_{\alpha_i\alpha_j} = \frac{\partial^2 L}{\partial\alpha_i\partial\alpha_j}. \tag{9}$$

In (9) only $(N+1-n)$ differentials are independent as $d\boldsymbol{\alpha}$ must satisfy

$$J(\boldsymbol{\alpha}_0)\,d\boldsymbol{\alpha} = \mathbf{0}, \tag{10}$$

where J is given by (4) and is of rank n because of the independence of the terminal constraint equations. System (10) is a direct consequence of the equality constraints $\mathbf{x}(\boldsymbol{\alpha}) = \mathbf{x}_T$. Using (10) and expressing n dependent differentials in terms of the independent differentials and substituting the result into (9) we obtain a quadratic form in restricted variables. If it turns out that this form is positive definite then $\boldsymbol{\alpha}_0$ is a local minimizer for Problem (P2).

Let us describe a computationally straightforward post-processing procedure that determines the sign of the quadratic form (9) under the constraints (10). Introduce the notation

$$H = [L_{\alpha_i\alpha_j}(\boldsymbol{\alpha}_0)] \quad \text{and} \quad \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{N+1})^T = (d\alpha_1, d\alpha_2, \ldots, d\alpha_{N+1})^T,$$

rewrite equations (9) and (10) in this notation, and consider

$$\boldsymbol{\beta}^T H \boldsymbol{\beta} \quad \text{subject to } J(\boldsymbol{\alpha}_0)\boldsymbol{\beta} = \mathbf{0}\,. \tag{11}$$

We assume that the conditions $\mathbf{x}(\boldsymbol{\alpha}) = \mathbf{x}_T$ are independent and therefore the rank of $J(\boldsymbol{\alpha}_0)$ is n. Let B be an $n \times n$ matrix formed by n linearly independent columns of $J(\boldsymbol{\alpha}_0)$ and $\hat{\boldsymbol{\beta}}$ be a vector made of the corresponding β_i. Similarly, let G be an $n \times \nu$ matrix formed by the remaining $\nu = N+1-n$ columns of $J(\boldsymbol{\alpha}_0)$ and $\tilde{\boldsymbol{\beta}}$ be a vector made of the components of $\boldsymbol{\beta}$ with the corresponding subscripts. Using the equivalence $J(\boldsymbol{\alpha}_0)\boldsymbol{\beta} = \mathbf{0} \;\Leftrightarrow\; B\hat{\boldsymbol{\beta}} = -G\tilde{\boldsymbol{\beta}}$, express $\hat{\boldsymbol{\beta}}$ in terms of $\tilde{\boldsymbol{\beta}}$ as $\hat{\boldsymbol{\beta}} = A\tilde{\boldsymbol{\beta}}$ where $A = -B^{-1}G$. Taking the permutation matrix P such that $\boldsymbol{\beta} = P[\hat{\boldsymbol{\beta}}^T \tilde{\boldsymbol{\beta}}^T]^T$ and substituting into the form in (11) we obtain

$$\boldsymbol{\beta}^T H \boldsymbol{\beta} = \left[\hat{\boldsymbol{\beta}}^T \tilde{\boldsymbol{\beta}}^T\right] P^T H P \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{\beta}} \end{bmatrix} = \tilde{\boldsymbol{\beta}}^T \left[A^T \middle| I_{\nu\times\nu}\right] P^T H P \left[\frac{A}{I_{\nu\times\nu}}\right] \tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}^T Q \tilde{\boldsymbol{\beta}},$$

our quadratic form in restricted variables. Positive-definiteness of this form is a sufficient condition for $\boldsymbol{\alpha}_0$ to be a local minimizer of $\phi(\mathbf{x}(\boldsymbol{\alpha}), \boldsymbol{\alpha})$.

Some numerical considerations

The hessian of the Lagrange function used in (9) and (6) can be written as

$$[L_{\alpha_i\alpha_j}] = \left(\frac{\partial \mathbf{h}}{\partial \alpha_1}, \ldots, \frac{\partial \mathbf{h}}{\partial \alpha_{N+1}}\right), \tag{12}$$

where $\mathbf{h}(\boldsymbol{\alpha}, \boldsymbol{\lambda}) := \nabla_{\boldsymbol{\alpha}}\phi + J^T\boldsymbol{\lambda}$. It turns out that it is quite acceptable to evaluate (12) using central difference quotients for the iterative procedure that uses (6). Care must be taken, however, if the hessian is needed to verify sufficient conditions for a minimum. In particular, the described method of evaluation for the hessian does not guarantee a symmetric matrix. Furthermore, the technique of computation of the Jacobian using (7) and (8) may turn out to be unsatisfactory at points where the hessian has large values. In such cases, once a moderate proximity to a possible solution has been reached, we could switch to a slower method of Jacobian evaluation using finite differences.

4. Example Applications

4.1. F–8 aircraft

The following mathematical model of the F–8 aircraft, which emanates from Garrard and Jordan [22], has been used in various control studies: [23, 7, 12, 15]

$$\begin{aligned}\dot{x}_1 =& -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 - x_1^2x_3 \\ &+ 3.846x_1^3 - 0.215u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3, \\ \dot{x}_2 =& x_3, \\ \dot{x}_3 =& -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967u \\ &+ 6.265x_1^2u + 46x_1u^2 + 61.4u^3,\end{aligned} \tag{13}$$

where x_1 is the angle of attack in radians, x_2 the pitch angle, x_3 the pitch rate in rad/s, and the control u is the tail deflection angle. In Kaya and Noakes [7] the setting is given for a special case, namely for bang–bang constrained controls, where *open-loop* time-optimal switchings are computed to stabilize the aircraft from an initial disturbance of 26.7 deg in the angle of attack x_1. The tail deflection angle has been assumed to be set at piecewise constant values, either 3 or -3 degrees, i.e. approximately $u(t) = 0.05236$ rad or -0.05236 rad. For $u(t)$ these approximate radian values will be used, while for $x_1(t)$ the exact value $26.7\pi/180$ rad is used, in the same manner as in Lee *et al.* [12] and Kaya and Noakes [15] in finding bang–bang constrained time-optimal controls. More specifically,

$$\mathbf{x}_0 = \frac{\pi}{180}(26.7,\, 0,\, 0)^T,\ \mathbf{x}_T = (0,\, 0,\, 0)^T,\ u = \pm 0.05236.$$

Since the Lagrange equations are the conditions for a local minimizer, computation results depend on the initial guess, which requires specification of both the arc-times and the Lagrange coefficients. We organize various initial guesses and corresponding results in separate examples.

First we demonstrate the working of our technique in detail with four example cases below, namely with 1-3. In particular, in Example 1 1 we also show how to check the sufficient conditions for the bang–bang constrained optimality. Then we give a comparison of the existing local minima and the two local minima presented in this paper. Finally we examine the best available switching structures, not only for $x_1(0) = 26.7$ deg, but also for other values of $x_1(0)$. We tabulate these structures and give comments.

Example 1. Consider a 5-arc configuration with $u(0) = 0.05236$ and initial guess $\xi_1 = 1$, $\xi_2 = 0.3$, $\xi_3 = 1.5$, $\xi_4 = 1$, $\xi_5 = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 0$. This leads to the solution $\xi_1 = 1.132765$, $\xi_2 = 0.347492$, $\xi_3 = 1.608881$, $\xi_4 = 0.692379$, $\xi_5 = 0$, $\lambda_1 = 2.322838$, $\lambda_2 = -1.396123$, and $\lambda_3 = -0.942077$. The obtained total time of 3.781517 is significantly smaller than the 6-arc solution of 5.742177 given in Kaya and Noakes [15] and the 4-arc solution of 6.035 in Lee *et al.* [12]. The program required 33 iterations to obtain the above arc-times and Lagrange coefficients, which satisfy equations (3) with an accuracy of order 10^{-6} (in terms of nearness to the target $(0, 0, 0)^T$). As one would expect, less accurate solutions can be achieved with fewer iterations. Also, note that the last arc time is of length zero, indicating that an optimal solution in fact only requires four arcs.

Now let us examine the sufficient conditions. The approximate value for the hessian is

$$H = 10^3 \times \begin{bmatrix} 2.119 & -0.247 & -0.132 & -0.059 & 0 \\ -0.247 & 0.037 & 0.018 & 0.003 & 0 \\ -0.132 & 0.018 & 0.010 & -0.001 & 0 \\ -0.059 & 0.003 & -0.001 & 0.004 & 0 \\ 0 & 0 & 0 & 0 & 0.004 \end{bmatrix}, \tag{14}$$

and the value of the Jacobian matrix (4) at $\boldsymbol{\alpha}_0$ corresponding to the arc-time vector $\boldsymbol{\xi}_0$ is

$$J(\boldsymbol{\alpha}_0) = \begin{bmatrix} -1.7032 & -1.3555 & -1.6892 & 0.0186 & 0 \\ 15.5437 & -3.7508 & -2.3257 & 0 & 0 \\ -24.9752 & 3.4679 & 1.9744 & 1.8123 & 0 \end{bmatrix} .$$

We take the first three (linearly independent) columns of $J(\boldsymbol{\alpha}_0)$ to form

$$B = \begin{bmatrix} -1.7032 & -1.3555 & -1.6892 \\ 15.5437 & -3.7508 & -2.3257 \\ -24.9752 & 3.4679 & 1.9744 \end{bmatrix} \quad \text{and therefore} \quad G = \begin{bmatrix} 0.0186 & 0 \\ 0 & 0 \\ 1.8123 & 0 \end{bmatrix}.$$

Since $\boldsymbol{\beta} = [\hat{\boldsymbol{\beta}}^T \tilde{\boldsymbol{\beta}}^T]^T$, the corresponding permutation matrix is the 5×5 identity matrix. Then

$$A = -B^{-1}G = \begin{bmatrix} 0.1990 & 0 \\ 1.8755 & 0 \\ -1.6946 & 0 \end{bmatrix},$$

and the matrix of the quadratic form in restricted variables is

$$Q = [A^T \ \operatorname{diag}(1,1)] I_{5\times5} H I_{5\times5} \begin{bmatrix} A \\ \operatorname{diag}(1,1) \end{bmatrix} \approx \begin{bmatrix} 28.1 & 0 \\ 0 & 4.0 \end{bmatrix}.$$

The positive definiteness of Q verifies that the solution found is indeed a local minimum.

Remark 1. If, during computations, an arc-time becomes sufficiently small, we can skip ODE numerical integration in the corresponding interval by merely making the endpoint of the arc coincide with its starting point. This effectively eliminates dependence of our output parameters on this arc-time and can cause errors in calculating the hessian of the Lagrange function. To avoid such errors we can resume the ODE integration in those intervals once we have entered the final stage of iterations. Let us illustrate this with example 1. The lower right element of the hessian in (14) is given by

$$L_{\alpha_5\alpha_5} = 2 + \mathbf{x}_{\alpha_5\alpha_5}^T \boldsymbol{\lambda} = 2 + (2\mathbf{x}_{\xi_5} + 4\alpha_5^2 \mathbf{x}_{\xi_5\xi_5})^T \boldsymbol{\lambda}.$$

Since $\alpha_5 = 0$ we have $L_{\alpha_5\alpha_5} = 2(1 + \mathbf{x}_{\xi_5}^T \boldsymbol{\lambda})$. Taking into account that

$$\mathbf{x}_{\xi_5} = \mathbf{f}_5(\mathbf{x}_T) = \begin{bmatrix} -0.215u + 0.63u^3 \\ 0 \\ -20.967u + 61.4u^3 \end{bmatrix}$$

and substituting $\boldsymbol{\lambda}$ from the above table we find $L_{\alpha_5\alpha_5} = 4$, which agrees with (14). If the dependence on ξ_5 had been excluded we would obtain the erroneous result $L_{\alpha_5\alpha_5} = 2$.

Example 2. Consider a 6-arc configuration with $u(0) = 0.05236$ and initial guess $\xi_1 = 1$, $\xi_2 = 1$, $\xi_3 = 1$, $\xi_4 = 1$, $\xi_5 = 1$, $\xi_6 = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 0$. This leads to the solution $\xi_1 = 1.1327648$, $\xi_2 = 0.3474915$, $\xi_3 = 1.6088814$, $\xi_4 = 0.2223491$, $\xi_5 = 0$, $\xi_6 = 0.4700298$, $\lambda_1 = 2.322838$, $\lambda_2 = -1.396123$, and $\lambda_3 = -0.942077$ with total time 3.781517. The combination of the last three arcs in this result is equivalent to one arc, as the middle arc is of zero length. If the number of arcs N were chosen larger, similar results are obtained, with a larger number of arcs of zero length. The sum of the remaining two arc-times has the same value as the value of the fourth arc-time in example 1. Though the same tolerance of 10^{-6} has been used for this example we provide more digits in final arc-time values to offset the effect of the rounding errors. The final arc configuration is reached in 29 iterations.

A different choice of initial arc-times and Lagrange multipliers can yield a different minimum time as illustrated in the following example.

Example 3. Consider a 6-arc configuration with $u(0) = 0.05236$ and initial guess $\xi_1 = 0.5$, $\xi_2 = 1$, $\xi_3 = 0.5$, $\xi_4 = 1$, $\xi_5 = 0.5$, $\xi_6 = 0.5$, $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 0$. This leads to the solution $\xi_1 = 0.102917$, $\xi_2 = 1.927923$, $\xi_3 = 0.166868$, $\xi_4 = 2.743384$, $\xi_5 = 0.329923$, $\xi_6 = 0.471162$, $\lambda_1 = 10.951790$, $\lambda_2 = -7.673815$, and $\lambda_3 = -1.030559$ with total time 5.742177. This result required 9 iterations. The arc times (and so the total time) are the same as those reported in Kaya and Noakes [15] for a similar configuration.

Known local minima for $x_1(0) = 26.7$ deg

In this paper, two new local minima are reported in addition to those previously reported [7, 12, 15]. These five local solutions are listed in Table I. In the table a '+' arc indicates $u = 0.05236$, and a '−' arc $u = -0.05236$. The results from References [7] and [12] have been refined for $x_1(0) = 26.7$ deg and an accuracy of 10^{-6} for the terminal state. Of the two four-arc solutions listed, the one with $t_f \approx 3.8$ is remarkably smaller than the rest of the local solutions, which all have $t_f > 5.7$. It is conceivable to think that there exist further local solutions.

	Arcs						Total
Ref.	+	−	+	−	+	−	time (t_f)
New		5.8198517	0.3173412	0.5859682			6.723161
[7]	0.0762850	5.4114404	0.3534928	0.5657006			6.406919
[12]		2.1885418	0.1639897	2.8811229	0.3296798	0.4719214	6.035256
[15]	0.1029158	1.9279261	0.1668660	2.7433790	0.3299230	0.4711621	5.742172
New	1.1327648	0.3474916	1.6088813	0.6923790			3.781517

Table I. F–8 aircraft and a list of local minima for $x_1(0) = 26.7$ deg.

Switching structures for different $x_1(0)$

So far, computations have been carried out to find controls that would stabilize the aircraft in minimum time for an initial angle of attack of 26.7 deg. A four-arc switching structure gives the best solution among the four identified structures. These identified structures are namely 3-, 4-, 5-, and 6-arc solutions. In fact two different local minima with four arcs have been found; but we count only the one that gave a smaller terminal time as the representative of the 4-arc structure.

It is interesting to note that the structure of the best available solution depends on the initial value of the angle of attack, $x_1(0)$. Table II lists best local minima for different values of $x_1(0)$. In each of these cases we set $x_2(0) = x_3(0) = 0$. We see that for $x_1(0) \geq 22.17$ the 4-arc structure gives the best solution. The 6-arc structure is the best for the interval $[19.83, 22.16]$. This is partly because it is not possible to find a feasible solution with 4-arcs as $x_1(0)$ approaches 22.16 from above - the second arc length becomes zero, effectively reducing the number of arcs to just two. For the interval $[18.07, 19.82]$ the best solution is given by the 5-arc structure. For smaller angles of attack, i.e. for the interval $[0, 18.06]$, the only local solution has three arcs.

The graph of the tabulated solutions of minimum terminal time vs angle of attack is given in Figure 2. While one observes a continuous transition from 3- to 5-arc structure, and then from 5- to 6-arc structure, there is a downward jump in the terminal time when the transition from the 6- to 4-arc structure occurs. For the 4-arc solutions it is interesting to note an initial decrease in t_f with increasing angle of attack. It may be noteworthy to point that the stall angle of the F–8 aircraft (i.e. the angle at which the aircraft loses aerodynamic lift) is reported to be around 23.5 deg [23].

$x_1(0)$ [deg]	Arcs +	−	+	−	+	−	Total time (t_f)
30.00	0.8940944	0.3961494	1.8155502	0.9469457			4.052740
29.00	0.9735248	0.4006068	1.7118241	0.8136131			3.899569
28.00	1.0425148	0.3824771	1.6560244	0.7475692			3.828585
27.00	1.1114472	0.3564054	1.6181225	0.7032858			3.789261
26.70	1.1327648	0.3474916	1.6088813	0.6923790			3.781517
26.00	1.1845867	0.3248125	1.5900577	0.6697746			3.769231
25.00	1.2666261	0.2870643	1.5682169	0.6422135			3.764121
24.00	1.3660058	0.2396005	1.5504999	0.6173140			3.773420
23.00	1.5067582	0.1701692	1.5346431	0.5899452			3.801516
22.17	1.8344286	0.0046401	1.4953115	0.5267910			3.861171
22.16	0.0297598	1.9835306	0.0837914	1.1767039	0.3149330	0.4829472	4.071666
22.00	0.0276203	1.9889547	0.0815167	1.1278587	0.3152130	0.4843825	4.025546
21.00	0.0146635	2.0229597	0.0681359	0.8345367	0.3191395	0.4921463	3.751581
20.00	0.0023450	2.0561571	0.0537879	0.5625374	0.3268068	0.4980022	3.499636
19.83	0.0003103	2.0618521	0.0509332	0.5181662	0.3286502	0.4988596	3.458772
19.82		2.0630338	0.0507151	0.5149202	0.3287948	0.4989214	3.456385
19.00		2.0455767	0.0364636	0.3450157	0.3391548	0.5019784	3.268189
18.07		2.0283485	0.0072670	0.1669932	0.3642955	0.5044789	3.071383
18.06		2.1917316	0.3702728	0.5073756			3.069380
18.00		2.1805695	0.3707050	0.5059541			3.057229
17.00		2.0048329	0.3737656	0.4864233			2.865022
15.00		1.6875197	0.3689474	0.4687992			2.525266
10.00		1.0059517	0.3921218	0.4544113			1.852485
5.00		0.5306211	0.4320906	0.3540284			1.316740

Table II. Best local minima for different values of $x_1(0)$.

4.2. Rayleigh problem

The Rayleigh problem arises from the so-called tunnel-diode oscillator, which is an electric circuit. Maurer and Oberle [25] look at the problem of finding optimal controls minimizing a

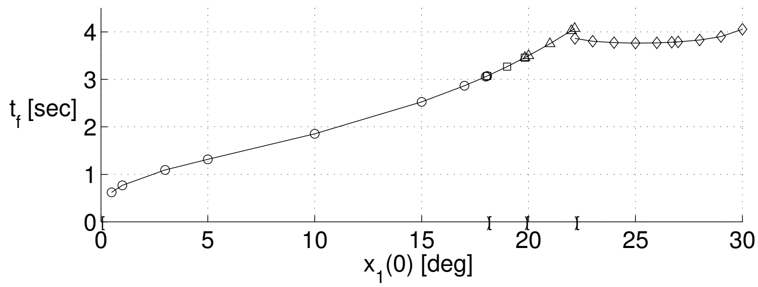

Figure 2. Minimum t_f vs angle of attack. '$\circ$' denotes a 3-arc solution, '$\square$' a 5-arc solution, '$\triangle$' a 6-arc solution, and '$\diamond$' a 4-arc solution.

mixed cost criterion with free time, while Maurer and Osmolovskii [26] concentrate on finding a time-optimal control. The problem with the mixed cost criterion is posed as follows. The objective is to minimize the functional $c\,t_f + \int_0^{t_f}(u^2 + x_1^2)dt$ subject to

$$\begin{aligned}
&\dot{x}_1 = x_2,\\
&\dot{x}_2 = -x_1 + x_2(1.4 - 0.14x_2^2) + 4u, \qquad |u(t)| \le 1\ ,\\
&\mathbf{x}_0 = (-5,\ -5), \quad \mathbf{x}_T = (0,\ 0),
\end{aligned}$$

where c is a positive constant. The state variable x_1 denotes a certain electric current, and the control u the voltage of the generator in the circuit. The time-optimal control that has been reported for this system is of bang–bang type [26]. The reported solution of the mixed-cost problem is continuous; however it roughly resembles a bang–bang profile. This motivates us to apply our method to find the bang–bang constrained time-optimal and mixed-cost controls for the Rayleigh problem. It may be desirable to switch between two constant values of the voltage in an electric circuit, rather then adjusting the value of the voltage continuously.

For our case we take $u(t) \in \{-1, 1\}$. Therefore the functional to minimize becomes

$$(1 + c)\,t_f + \int_0^{t_f} x_1^2\,dt\ . \tag{15}$$

For large values of c, the solution is close to being time-optimal. However in obtaining the time-optimal solution we consider minimizing t_f directly.

Consider minimizing (15) first. Let $c = 1/16$ as in Reference [25].

Consider a 4-arc configuration with $u(0) = 1$ and initial guess $\xi_1 = 1.5$, $\xi_2 = 2$, $\xi_3 = 1$, $\xi_4 = 0.5$, $\lambda_1 = 0$, and $\lambda_2 = 0$. This leads to the solution $\xi_1 = 1.4761410$, $\xi_2 = 1.7606912$, $\xi_3 = 1.7606912$, $\xi_4 = 0$, $\lambda_1 = 0.5831380$, and $\lambda_2 = -0.2656250$ with total time 3.773841 and total cost 45.746978. The solution is found in 14 iterations. The optimal solution appears to involve 3 arcs regardless of initial guess.

Table III lists solutions obtained for different values of c in (15). Note that the case indicated by $c \longrightarrow \infty$ corresponds to a time-optimal control; however we minimize t_f directly in the implementation of our method. Also note that when $c = -1$ in (15) one has the problem of minimizing the integral cost $\int_0^{t_f} x_1^2\,dt$ alone.

c	Arcs			Total time (t_f)	Total cost
	+	−	+		
∞	1.1205068	2.1895401	0.3581265	3.668173	
1/16	1.4761410	1.7606912	0.5370092	3.773841	45.746978
0	1.4785180	1.7581199	0.5386684	3.775306	45.511067
-1	1.5208743	1.7130176	0.5691841	3.803076	41.722599

Table III. Optimal bang–bang constrained solutions with different values of c for the Rayleigh problem.

5. Conclusion

In this paper a numerical algorithm to solve a class of bang–bang constrained optimal control problems has been presented. The technique has been applied to a number of example control systems. Second-order sufficient conditions have also been devised and their use demonstrated in some typical cases. A comprehensive set of numerical experiments have been carried out for the time-optimal switching structure for stabilizing the F–8 aircraft.

Our technique is at present devised for single input systems. A two-input version of the STC method [17] may lay the necessary ground work for an extension to multi-input systems.

In gradient calculations, the concatenation of arcs in a continuous manner resembles the simple shooting method. A multiple shooting scheme for the same calculations would possibly increase the efficiency of the algorithm. A multiple-shooting scheme may first be implemented in STC which uses the same style of gradient calculations.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge partial support from an Australian Research Council Small Grant. Helpful discussions with Helmut Maurer and his advice for the inclusion of the Rayleigh problem as an example application is deeply appreciated. We thank the referees for constructive advice in revising this paper.

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