OPTIMAL STRATEGIES FOR THE PROGRESSIVE MONTY HALL PROBLEM

STEPHEN K. LUCAS, JASON ROSENHOUSE

1. Introduction

In the classical Monty Hall problem you are a contestant on a game show confronted with three identical doors. One of them conceals a car while the other two conceal goats. You choose a door, but do not open it. The host, Monty Hall, now opens one of the other two doors, careful always to choose one he knows to conceal a goat. You are then given the options either of sticking with your original door, or switching to the other unopened door. What should you do to maximize your chances of winning the car?

Most people find the correct solution, that you double your chances of winning by switching doors, to be counterintuitive. Also counterintuitive is the fact that if Monty selects his door at random and just happens to choose one with a goat, then there is no longer any advantage to be gained from switching. A lucid explanation of these points can be found in [5].

Several n door versions of the problem are known. One of them, which we call the Progressive Monty Hall problem, starts with $n > 3$ doors. One door conceals a car, while the other $n-1$ doors conceal goats. You select one door, but do not open it. Monty chooses at random one of the other doors that he knows conceals a goat, and opens it. You are then given the option of switching to a different door. After making your choice, Monty reveals another goat and again offers you the option of switching. This continues until only two doors remain. You make your final choice, and receive whatever is behind your door. What strategy maximizes your chances of winning the car?

The case $n = 4$ was solved by Rao and Rao [7]. By enumerating the sample space, they established that the best strategy is to stick with your original door until only two doors remain, and then switch. They asserted, but did not prove, that this strategy is optimal for all $n \geq 4$, with the probability of success being $(n-1)/n$. Using methods based on Bayes' theorem and conditional probabilities, the optimality of this strategy was proved by the authors (in joint work with A. Schepler) in [6].

Here we present a different approach. Using recurrence relations, we prove that by following the strategy of switching every time you win with probability $1 - 1/e$. We also establish the optimal strategy to follow when you are determined to switch a given number k times. We then present another proof of the optimality of the "switch at the last minute" strategy.

2. Switching Every Time

It has been our experience in presenting this problem to students that the strategy of switching doors at every opportunity is invariably popular. This discussion typically comes after a long struggle to persuade them of the benefits of switching in the classical version. The take home message seems to be that switching is a very good thing indeed, which might explain the popularity of this approach. In this section we will prove that your probability of success with this strategy approaches $1 - 1/e$ as $n \to \infty$.

We assume that every time Monty reveals a goat we select randomly from among the unopened doors different from our current choice. Denote by a_n the probability of winning with this strategy. Our analysis now splits into two cases, depending on whether our initial choice is correct or incorrect. Let b_n denote the probability of winning if we begin with n doors and our initial choice conceals a goat, and let c_n denote the probability of winning if we begin with n doors and our initial choice conceals the car. Since there is one car and $n-1$ goats, we can write

$$a_n = \left(\frac{n-1}{n}\right) b_n + \left(\frac{1}{n}\right) c_n. \tag{1}$$

Suppose our initial choice conceals a goat. Monty opens a door and reveals a goat. This time, among the remaining $n-2$ doors different from our current choice, there is one that conceals the car and $n-3$ that conceal goats. Thus, there is a probability of $1/(n-2)$ that we will switch to the car, and a probability of $(n-3)/(n-2)$ that we will switch to a goat. Consequently, we can write

$$b_n = \left(\frac{n-3}{n-2}\right) b_{n-1} + \left(\frac{1}{n-2}\right) c_{n-1}. \tag{2}$$

Alternatively, suppose our current choice conceals the car. Again Monty reveals a goat. In this case each of the remaining $n-2$ doors will conceal a goat. It follows that we will switch to a goat, and therefore

$$c_n = b_{n-1}. \tag{3}$$

We can use equation (3) to eliminate the c_i terms in equations (1) and (2). We obtain

$$a_n = \left(\frac{n-1}{n}\right) b_n + \left(\frac{1}{n}\right) b_{n-1} \tag{4}$$

and

$$b_n = \left(\frac{n-3}{n-2}\right) b_{n-1} + \left(\frac{1}{n-2}\right) b_{n-2}. \tag{5}$$

Note that $b_2 = 0$ and $b_3 = 1$.

Repeated applications of equation (5) leads to

$$\begin{aligned} b_{n+2} &= \left(\frac{n-1}{n}\right) b_{n+1} + \left(\frac{1}{n}\right) b_n \\ &= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1} b_n + \frac{1}{n-1} b_{n-1}\right) + \left(\frac{1}{n}\right) b_n \\ &= \left(\frac{n-1}{n}\right) b_n + \left(\frac{1}{n}\right) b_{n-1}. \end{aligned}$$

Since this last expression appears as the right-hand side of (4), we see that $b_{n+2} = a_n$. If we now rewrite (5) with $n+2$ in place of n and then use this substitution, we get the recurrence

$$a_n = \left(\frac{n-1}{n}\right) a_{n-1} + \left(\frac{1}{n}\right) a_{n-2}, \tag{6}$$

with $a_0 = 0$ and $a_1 = 1$.

To solve (6), we rewrite it as

$$a_n - a_{n-1} = -\frac{1}{n}(a_{n-1} - a_{n-2}).$$

Since $a_1 - a_0 = 1$, we have

$$a_n - a_{n-1} = -\frac{(-1)^n}{n!},$$

which implies that the probability of winning is

$$a_n = a_n - a_0 = \sum_{i=1}^{n}(a_i - a_{i-1}) = -\sum_{i=1}^{n}\frac{(-1)^i}{i!}.$$

It follows that

$$\lim_{n\to\infty} a_n = -\sum_{i=1}^{\infty}\frac{(-1)^i}{i!} = 1 - \sum_{i=0}^{\infty}\frac{(-1)^i}{i!} = 1 - \frac{1}{e},$$

recognizing that the infinite series is that for e^x with $x = -1$. The number of doors n does not need to be very large before the probability stabilizes at 0.632.

3. Other Things With Probability $(1 - 1/e)$

Perhaps it is a bit surprising that the number e should appear in a probability problem based on a game show. Making it even more remarkable is the prevalence of $1 - 1/e$ as the asymptotic probability of a variety of quite different phenomena.

- A *derangement* [8] is a permutation of the integers from one to N such that for all $1 \leq i \leq N$ the number i does not appear in the i-th place. The probability that a permutation is not a derangement approaches $1-1/e$ as N approaches infinity. The two most common proofs are by application of the inclusion-exclusion principle, or by developing a recurrence relation very similar to equation (6).
- Suppose you have n numbered balls. You select one at random, jot down its number, then replace it. You repeat this procedure n times. Since at every step all n balls are available, the probability of not selecting a given ball at each drawing is $1 - 1/n$.

After making n selections, the probability of a given ball never having been chosen is $(1 - 1/n)^n$. Given the definition

$$e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n,$$

we see that as n increases, the probability that a given ball is never selected in n drawings is $1/e$. Thus, the probability of selecting a given ball at least once in n drawings approaches $1 - 1/e$.

This result is important in bootstrapping methods in statistics, where we want to estimate properties of a population based upon sampling from an approximate distribution, as in [1]. A particular example is known as 0.632 bootstrapping, where $0.632 \approx 1 - 1/e$.

- Imagine that we are interviewing n applicants for a job. We know that there exists a unique ordering of the candidates from best to worst, but we do not know what that ordering is. We can only assign relative rankings as we interview each candidate. At the end of each interview we must make a decision as to whether to accept or reject each candidate. Once rejected, we are not allowed to recall a candidate. Our goal is to hire the best candidate; hiring the second best is the same as hiring the worst for our purposes. What strategy will maximize our chances of hiring the best candidate?

  This is variously known as the secretary problem, the Sultan's dowry problem, the marriage problem, or the best choice problem. Following Havil [3], the optimal strategy begins by rejecting the first k applicants, where $k = \lfloor n/e \rfloor$. We then hire the next applicant who is superior to all those who came before. If no such applicant exists, then we agree to hire the last candidate even though that implies failure. It turns out that following this strategy you will fail to hire the best candidate with probability approaching $1 - 1/e$ an n increases. The result is related to approximating the harmonic series by a natural logarithm, which introduces e.

What makes these examples even more fascinating is that we have been unable to find any comparable problem (apart from the obvious cases involving the exponential probability distributions) where the probability does not involve $1/e$. There are many probability problems related to that other classic constant π, but the familiar examples all provide different algebraic expressions. For example, the probability that two randomly chosen positive integers are relatively prime is $6/\pi^2$, as is the probability that a randomly chosen integer is square-free [2]. The probability that the triple $(x, y, 1)$, where $0 < x, y < 1$ represents the lengths of the sides of an obtuse triangle is $(\pi - 2)/4$ [4]. The solution to the Buffon needle problem (in which a needle of length L is tossed at random onto a plane ruled with parallel lines at distance $d \geq L$ from each other, and we seek the probability that the needle intersects a line) is given by $2L/(\pi d)$ [5].

4. Other Strategies

What can be said on behalf of other strategies? We note first that equation (1), though formulated with the "Switch Every Time" strategy in mind, is actually valid for any strategy. More specifically, let S be a given strategy and denote by a_n the probability of winning with S at the moment when n doors remain in play. We now make our initial door choice. If b_n denotes the probability of winning with S given that our current choice conceals a goat, and c_n denotes the probability of winning given that our current choice conceals the car, then a_n, b_n and c_n are related via equation (1).

Furthermore, since it is assumed that the doors are identical, and therefore that their numbering is arbitrary, we only need to consider strategies that call for switching at specific moments during the game. Strategies such as, "Switch if Monty opens an even door, but stick otherwise," can not be optimal, and we will not consider them.

Let us assume, then, that our strategy calls for us to switch doors a total of k times with $k \leq n - 2$. We will assume that we choose our new door randomly from the available options each time we switch. Denote by $\{m_i\}_{i=1}^{k}$ the number of doors remaining when we make the $k - i + 1$-st switch. We have $3 \leq m_1 < m_2 < \cdots < m_k \leq n$.

For any integer j, the manner in which the probabilities b_j and c_j are related to b_{j-1} and c_{j-1} will depend on whether or not we switch at the moment when j doors remain. If we switch, then the probabilities are related in the manner described by equations (2) and (3). If we do not switch, then the probabilities do not change. To see this, note that our probability of winning by sticking with our present door is equal to the probability that it conceals the car. A straightforward argument using Bayes' theorem shows that this probability can not change so long as we maintain this door as our selection. Consequently, our probability of winning can change only at those moments of the game when we decide to switch doors.

It follows that we have

$$b_{m_i} = \left(\frac{m_i - 3}{m_i - 2}\right) b_{m_{i-1}} + \left(\frac{1}{m_i - 2}\right) c_{m_{i-1}} \quad \text{and} \quad c_{m_i} = b_{m_{i-1}}, \tag{7}$$

for all $1 \leq i \leq k$. For any subscript $j \neq m_i$ for any i, we have $b_j = b_{j-1}$ and $c_j = c_{j-1}$.

To simplify the notation, we define $\beta_i = b_{m_i}$ and $q_i = m_i - 2$. This leads to

$$\beta_i = \left(\frac{q_i - 1}{q_i}\right) \beta_{i-1} + \left(\frac{1}{q_i}\right) \beta_{i-2}, \tag{8}$$

for $i = 1, 2, \dots, k$. Note that we have the initial conditions $\beta_{-1} = 1$ and $\beta_0 = 0$. The probability of winning given there are n doors and a given set of k door changes is thus

$$a_n = \left(\frac{n-1}{n}\right) \beta_k + \left(\frac{1}{n}\right) \beta_{k-1}. \tag{9}$$

To solve (9), set $\gamma_i = \beta_i - \beta_{i-1}$. Subtracting β_{i-1} from both sides of equation (8) then leads to

$$\gamma_i = \frac{-\gamma_{i-1}}{q_i},$$

with $\gamma_0 = -1$. Thus $\gamma_1 = 1/q_1$, $\gamma_2 = -1/(q_1 q_2)$, $\gamma_3 = 1/(q_1 q_2 q_3)$, and in general

$$\gamma_j = \frac{(-1)^{j+1}}{\prod_{i=1}^{j} q_i}.$$

Since $\beta_i = \gamma_i + \beta_{i-1}$,

$$\beta_1 = \frac{1}{q_1}, \quad \beta_2 = \frac{1}{q_1} - \frac{1}{q_1 q_2}, \quad \beta_3 = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3},$$

and in general

$$\beta_j = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \cdots + \frac{(-1)^{j+1}}{\prod_{i=1}^{j} q_i}. \tag{10}$$

Finally, substitution back in (9) gives us

$$a_n = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \cdots + \frac{(-1)^{k+1}}{\prod_{i=1}^{k} q_i} - \frac{(-1)^{k+1}}{n \prod_{i=1}^{k} q_i}. \tag{11}$$

Let us use (11) to analyze a few simple cases. If we never change doors, then $k = 0$, $\beta_0 = 0$ and $\beta_{-1} = 1$. It follows that $a_n = 1/n$. This makes sense. Our initial door choice is correct with probability $1/n$ and this probability can not change so long as it remains our choice (as can be shown with Bayes' theorem, for example). Note that as n increases, the probability of winning approaches zero.

If we change doors exactly once then $k = 1$ and

$$a_n = \frac{1}{q_1} - \frac{1}{n q_1} = \frac{n-1}{n q_1}.$$

The probability of winning is maximized, given our constraints on q_1, by choosing $q_1 = 1$, which is equivalent to $m_1 = 3$. This corresponds to switching at the last possible moment. Again, this makes sense. If you are only going to switch one time you should do so after Monty opens his final door. This strategy gives a probability of winning $a_n = (n-1)/n$, which approaches one as n increases.

If we change doors twice, $k = 2$ and

$$a_n = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{n q_1 q_2} = \frac{n(q_2 - 1) + 1}{n q_1 q_2}.$$

This is clearly maximized by choosing $q_1 = 1$ and $q_2 = n - 2$. The optimal strategy of changing doors immediately then leaving the final change to the last moment has probability $a_n = (n^2 - 3n - 1)/(n^2 - 2n)$, which also approaches one as n increases. Note, however, that for a given number of doors n, the best one change winning probability is better than that for two changes.

Finally, if we change doors three times, then $k = 3$ and

$$a_n = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \frac{1}{n q_1 q_2 q_3}.$$

Since $1/q_1$ is a common factor to all terms, we see that a_n is maximized by minimizing q_1. That is, we set $q_1 = 1$. Determining the appropriate values of q_2 and q_3 is trickier. The value of a_n increases with q_2 (suggesting that q_2 should be maximized), but decreases with q_3 (suggesting that q_3 should be minimized). We must balance these considerations with the fact that $q_2 < q_3$. This is accomplished by setting $q_3 = q_2 + 1$. With this substitution, it is straightforward to show that we should take $q_2 = n - 3$ and $q_3 = n - 2$. This corresponds to making the first two switches immediately, and then waiting until the end to make the third. This leads to a probability of success of

$$a_n = \frac{n^3 - 6n^2 + 9n - 1}{n(n-2)(n-3)}.$$

This approaches one as n increases, but is nonetheless a lower chance of success than in the two switch strategy.

Based on these examples, you might suspect that if you are determined to switch exactly $k > 2$ times, your best strategy is to make your first $k - 1$ switches as soon as possible, and then wait until the last possible minute to make your final switch. That suspicion is correct, as the following theorem shows:

Theorem 1. *Let S be a strategy for the Progressive Monty Hall problem that calls for you to switch exactly k times, with $2 \leq k \leq n - 2$. Then the probability of winning with S is maximized by making switches $1, 2, 3, \cdots, k - 1$ when there are $n - 1, n - 2, \cdots, n - k + 1$ doors remaining, respectively, and making the k-th switch when only two doors remain.*

Proof. The quantity to be maximized is a_n from equation (11), subject to the constraints that $q_i \in \mathbb{Z}$ for all i and

$$1 \leq q_1 < q_2 < \cdots < q_k \leq n - 2.$$

Equation (11) can be rewritten as

$$a_n = \frac{1}{q_1}\left(1 - \frac{1}{q_2}\left(1 - \frac{1}{q_3} + \frac{1}{q_3 q_4} + \cdots + \frac{(-1)^{k+1}}{\prod_{i=3}^{k} q_i} - \frac{(-1)^{k+1}}{n\prod_{i=3}^{k} q_i}\right)\right).$$

It is clear that a_n is maximized by setting $q_1 = 1$. This corresponds to making your final switch at the last possible moment.

To simplify the notation, set

$$\gamma_k = 1 - \frac{1}{q_3} + \frac{1}{q_3 q_4} + \cdots + \frac{(-1)^{k+1}}{\prod_{i=3}^{k} q_i} - \frac{(-1)^{k+1}}{n\prod_{i=3}^{k} q_i}.$$

Maximizing a_n now requires that we minimize γ_k/q_2.

To do this, first note that γ_k is an alternating series whose terms are strictly decreasing in magnitude. It follows that $\gamma_k \geq 1 - (1/q_3)$. Since $q_3 > 1$, this implies that $\gamma_k > 0$.

Next, view the function $f(q_1, q_2, \cdots, q_k) = \gamma_k/q_2$ as a function from $\mathbb{R}^k \to \mathbb{R}$; that is, allow the q_i's to be real-valued instead of integer-valued. Then we have

$$\frac{\partial f}{\partial q_2} = \frac{\gamma_k}{(q_2)^2} > 0,$$

for all values of q_2. It follows that γ_k/q_2 will be minimized when q_2 is maximized. With our constraints, that means setting $q_2 = n - k$.

But this, in turn, forces us to set $q_i = n - k + (i - 2)$ for $3 \leq i \leq k$. This corresponds to the strategy laid out in the theorem, and the proof is complete. □

5. Optimal door choices

Finally, Lucas *et al.* [6] proved that the best strategy for winning the progressive Monty Hall problem is changing doors once at the last possible moment – in the above notation, $k = 1$, $q_1 = 1$, and $a_n = 1 - 1/n$. The proof used an inductive argument based upon Bayes' Theorem and conditional probabilities. We can now use (11) to establish this fact in a more straightforward manner:

Theorem 2. *The optimal strategy for the progressive Monty Hall problem is to change doors once at the moment when only two doors remain in play.*

$k \backslash n$	3	5	10	20	50	100	200	500
1	0.6667	0.8000	0.9000	0.9500	0.9800	0.9900	0.9950	0.9980
2		0.7333	0.8875	0.9472	0.9896	0.9899	0.9950	0.9980
3		0.6333	0.8732	0.9443	0.9792	0.9898	0.9949	0.9980
4			0.8545	0.9410	0.9787	0.9897	0.9949	0.9980
5			0.8291	0.9373	0.9783	0.9896	0.9949	0.9980
8			0.6321	0.9226	0.9767	0.9892	0.9948	0.9980
18				0.6321	0.9697	0.9880	0.9945	0.9979
48					0.6321	0.9811	0.9935	0.9978
98						0.6321	0.9903	0.9975
198							0.6321	0.9967
498								0.6321

TABLE 1. **Probabilities of winning given k switches and n doors (to four digits accuracy)**

Proof. We begin by assuming that we change doors exactly k times, where $k > 1$. We have already shown that the $k = 1$ case (one switch at the end) is superior to the best $k = 2$ or 3 cases. For $k > 3$ and using the optimum strategy just proven, the probability of winning is

$$a_n = 1 - \frac{1}{(n-k)} + \frac{1}{(n-k)(n-k+1)} - \cdots + \frac{(-1)^k}{(n-k)(n-k+1)\cdots(n-2)} - \frac{(-1)^k}{n(n-k)(n-k+1)\cdots(n-2)}.$$

This is an alternating series where each term is strictly smaller in magnitude than its predecessor, and so has an upper bound

$$a_n < 1 - \frac{1}{(n-k)} + \frac{1}{(n-k)(n-k+1)} = 1 - \frac{1}{(n-k+1)} < 1 - \frac{1}{n}.$$

Thus, the optimum when changing doors more than once is an inferior strategy to changing doors at the last possible moment, completing the proof. □

We conclude with table 1 indicating actual probabilities of winning the car using the optimum strategies with k switches and n doors.

REFERENCES

[1] B. Efron and R. Tibshirani, *An Introduction to the Bootstrap*, Chapman and Hall, New York (1993).

[2] G. Hardy and E. Wright, *An Introduction to the Theory of Numbers, 5th ed.*, Clarendon Press, Oxford, England (1979) pp. 267-270.
[3] J. Havil, *Gamma*, Princeton University Press, Princeton and Oxford (2003) 134-138.
[4] Ross Honsberger, *Ingenuity in Mathematics*, Mathematical Association of America (1970) 3-6.
[5] Richard Isaac, *The Pleasures of Probability*, Springer-Verlag, New York (1995) 1-11, 24-27.
[6] S.K. Lucas, J. Rosenhouse and A. Schepler, The progressive Monty Hall problem, *submitted for publication* (2007).
[7] V.V. Bapeswara Rao and M. Bhaskara Rao, A three-door game show and some of its variants, *The Mathematical Scientist* **17**(2) 89-94 (1992).
[8] E.W. Weisstein, Derangement – From MathWorld, A Wolfram Web Resource, `http://mathworld.wolfram.com/Derangement.html`.

Department of Mathematics and Statistics, James Madison University, Harrisonburg, VA 22807, USA
E-mail address: `lucassk@jmu.edu`

Department of Mathematics and Statistics, James Madison University, Harrisonburg, VA 22807, USA
E-mail address: `rosenhjd@jmu.edu`