On the contact of a rigid sphere and a plate

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Abstract

We consider the problem of the contact between a rigid sphere and a thin initially flat plate. After reviewing some plate theory, we establish that a deformation where a finite piece of the plate takes the shape of the sphere is physically unrealisable, and that the contact region must be a ring. However, for small deflections using classical theory and looking at some typical parameter values, we find that the radius of the ring is so small that for practical purposes it should be considered as a point load. We also outline the case for large deflections.
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1 Introduction

At the February 2000 meeting of the Mathematics in Industry Study Group (MISG) in Adelaide, South Australia, one of the problems presented involved a lens fracture test, where a spectacle lens is put under an applied load through a spherical indentor made of steel (see Lucas & Hill [1]). If the lens deflects too far, or breaks, then it fails the test. The purpose of this paper is to outline some results obtained for the idealised problem of contact between a rigid sphere and a thin flat plate.

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While there is an enormous amount of literature on problems involving plates and shells, and many excellent texts, such as Timoshenko & Woinowsky-Krieger [2] and Szilard [3], there seems to be relatively little relevant material on contact or obstacle problems. A closely related problem can be found in Essenburg [4], where the solution for the deformation of a clamped circular plate by a paraboloid of revolution is derived. However, the classical plate equations are adjusted to allow for transverse shear deformation, and while the paper indicates that far better results are obtained than those using classical theory, it is beyond the scope of this paper. Westbrook [5] reformulates obstacle problems for beams and plates as variational inequalities, and solves them using finite elements. While classical theory is used, the problem addressed is one where the beam or plate is deformed by a given force, with a rigid barrier that may impede its deformation. We are more interested in the case where the applied force is not known *a priori*, but is determined as a function of the position of the indenting sphere. Finally, Yau & Gao [6] in a sense extend Westbrook's work (although completely independently) by developing a nonlinear variational inequality for the obstacle problem using the von Kármán equations for thin plates where the deformation is large compared to the plate thickness. They discuss uniqueness and existence of solutions, but do not develop numerical techniques based on their methods.

There has also been some work done on the indentation of thick plates. Keer and Miller [7] combine an infinite layer solution with plate bending theory. They use the boundary condition that there is no stress on the lower surface of the plate, which is true only for very thick plates. Tsai [8] considers the indentation of a thick plate supported by a rigid foundation, using the solution for the indentation of an elastic half-space (in effect, an infinitely thick plate). However these methods are not applicable to the indentation of a plate which has a thickness measured in millimeters.

2 Classical Plate Theory

2.1 Governing Equations

A plate is a flat noncurved solid whose thickness is at least an order of magnitude smaller than the smallest of its other dimensions. Its middle surface is defined as a surface that bisects the plate in its thickness dimension. We assume that the material of the plate is elastic, homogeneous and isotropic, the plate is of equal thickness throughout, and that deflections are small compared to plate thickness. Typically we allow deformations up to 1/5 of the thickness of the plate. We also assume that the slope of the deformed middle surface is small, that straight lines initially normal to the middle surface remain straight and normal to the middle surface (ie transverse shear is neglected), that stresses normal to the middle surface are negligible, and that the deflection of the plate is produced by the displacement of points of the middle surface normal to its initial position.

Under these conditions, the linear theory of elasticity can be used to derive the governing differential equation for a plate subjected to lateral loads (see, for example [3, §1.2], which we follow in the discussion of this section). This is known as classical plate theory, and the governing equation is

$$D\nabla^2\nabla^2 w = p_z, \tag{1}$$

where w is the deflection of the plate under the lateral load p_z, and ∇^2 is the Laplacian operator. It is usual to assume that the initially flat middle surface of the plate lies in the x-y plane, and deflection is positive in the negative z direction. The constant $D = Eh^3/12(1-\nu^2)$ is known as the flexural rigidity of the plate, with Young's modulus E, Poisson's ratio ν, and thickness h. Equation (1) is a fourth order nonhomogeneous partial differential equation whose solution can often be found analytically. The bending moments acting on the plate are given in terms of w as

$$m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \quad \text{and} \quad m_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right). \tag{2}$$

For a circular plate with axisymmetric loading and boundary conditions, and with polar coordinates (r,θ), w is independent of θ, and (1) in terms of the Laplacian operator $\nabla_r^2 = \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr}$ is the ordinary differential equation

$$\nabla_r^2\nabla_r^2 w(r) = \frac{d^4 w}{dr^4} + \frac{2}{r}\frac{d^3 w}{dr^3} - \frac{1}{r^2}\frac{d^2 w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{p_z(r)}{D}. \tag{3}$$

The bending moments in this case are

$$m_r = -D\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right), \quad \text{and} \quad m_\theta = -D\left(\nu\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right). \tag{4}$$

A variety of physically realistic boundary conditions can be posed for plate problems. The two we will be interested in pursuing here for circular plates are the simple support conditions,

$$w(r)|_{r=r_0} = 0, \quad \text{and} \quad m_r(r)|_{r=r_0} = -D\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)\bigg|_{r=r_0} = 0, \tag{5}$$

which are equivalent to the plate sitting on a rigid support at radius r_0 with deflection and bending moment zero, and the clamped edge condition,

$$w(r)|_{r=r_0} = 0, \quad \text{and} \quad \left.\frac{dw(r)}{dr}\right|_{r=r_0} = 0, \tag{6}$$

which sets the deflection and slope of the plate at the edge to zero.

The governing equation (3) with one of (5) or (6) can often be solved analytically for a particular load p_z. Using linearity, the solution can be written as the sum of homogeneous

(w_h) and particular (w_p) solutions. The solution to the homogeneous ode $\nabla_r^2 \nabla_r^2 w_h = 0$ can be written as $w_h = C_1 + C_2(r/r_0)^2 + C_3 \ln(r/r_0) + C_4 r^2 \ln(r/r_0)$. For the deflections and moments at the center of the plate to be well behaved, we require $C_3 = C_4 = 0$. For the case of an annular plate, these constants would be set by adding additional boundary conditions at the inner edge of the annulus. The particular solution w_p for a circular plate can be found by direct integration of (3) as

$$w_p(r) = \frac{1}{D} \int_0^r \frac{1}{\delta} \left\{ \int_0^\delta \gamma \left[\int_0^\gamma \frac{1}{\beta} \left(\int_0^\beta p_z(\alpha) \alpha \, d\alpha \right) d\beta \right] d\gamma \right\} d\delta. \tag{7}$$

In the case of $p_z(r)$ being either a point load at the center of the plate, or a ring load at some radius r_1, the deflections can be found in similar ways, with the discontinuities in the third derivative depending on the total load, and the boundary conditions giving the value of the constants. For a point load of strength P at the center of the plate with simple support and clamped edges, the deflections of the middle surface are

$$w(r) = \frac{P r_0^2}{16\pi D} \left[\frac{3+\nu}{1+\nu} \left(1-\rho^2\right) + 2\rho^2 \ln \rho \right] \text{ and } w(r) = \frac{P r_0^2}{16\pi D} \left(1 - \rho^2 + 2\rho^2 \ln \rho \right), \tag{8}$$

respectively, with $\rho = r/r_0$. For a ring load of total load P at radius r_1, the deflection with simple support is

$$w(r) = \begin{cases} \dfrac{P}{8\pi D} \left[\left(r_1^2 + r^2\right) \ln \dfrac{r_1}{r_0} + \left(r_0^2 - r_1^2\right) \dfrac{(3+\nu) r_0^2 - (1-\nu) r^2}{2(1+\nu) r_0^2} \right], r \le r_1, \\ \dfrac{P}{8\pi D} \left[\left(r_1^2 + r^2\right) \ln \dfrac{r}{r_0} + \left(r_0^2 - r^2\right) \dfrac{(3+\nu) r_0^2 - (1-\nu) r_1^2}{2(1+\nu) r_0^2} \right], r \ge r_1. \end{cases} \tag{9}$$

The same case with clamped edges is

$$w(r) = \begin{cases} \dfrac{P}{16\pi D} \left[\left(1 - \dfrac{r_1^2}{r_0^2}\right) \left(r_0^2 + r^2\right) + 2\left(r_1^2 + r^2\right) \ln \dfrac{r_1}{r_0} \right] \quad r \le r_1, \\ \dfrac{P}{16\pi D} \left[\left(1 - \dfrac{r^2}{r_0^2}\right) \left(r_0^2 + r_1^2\right) + 2\left(r_1^2 + r^2\right) \ln \dfrac{r}{r_0} \right], r \ge r_1. \end{cases} \tag{10}$$

2.2 Spherically Prescribed Deflection

We are interested in finding the deformed shape of a circular plate due to contact with a rigid sphere at a given position, and also in finding the required load p_z, since $2\pi \int_0^{r_0} p_z(r) r \, dr$ is the total force with which the sphere pushes on the plate. The technique we anticipate using is common to many contact problems: assume a certain proportion of the sphere is in contact with the plate. In the contact region, the deformation is the shape of the sphere. There is no load on the rest of the plate, and so a solution can be found which takes into account the edge boundary conditions. We then require continuity of the deformation and its first and second derivatives at the point where the sphere leaves the plate. These conditions cannot in general be satisfied, and in principle we end up with a single nonlinear equation for the proportion of the sphere that contacts the plate. Once this proportion has been found, p_z and the total load on the plate can be calculated. For a much more general discussion on contact-impact problems and contact regions within the domain of finite element approximation methods, see Zhong [9].

Assume that the undeformed plate is in the x-y plane, and the deformation w of the middle surface is measured as positive in the negative z direction. The plate is deformed by a sphere of radius R and center $(x, y, z) = (0, 0, c)$ (we require $R - c > 0$ so that the sphere actually contacts the plate). If r_1 is the radius of the region of contact between the plate and the sphere, then the deformation of the top surface of the plate (half the height of the plate above the middle surface) is $w(r) - h/2 = -c + \sqrt{R^2 - r^2}$ for $0 \leq r \leq r_1$. Note that the geometry requires $r_1 < R$, and the minus sign in front of the c is due to w being positive down. Then, from (3) on $[0, r_1]$, the load due to the spherical deformation is

$$\frac{p_z(r)}{D} = \frac{r^4 - 8R^2r^2 - 8R^4}{(R^2 - r^2)^{7/2}}. \tag{11}$$

An immediately obvious problem is that $p_z(0) = -8/R^3 < 0$. The real positive solution to $r^4 - 8R^2r^2 - 8R^4 = 0$ is $r = \sqrt{4 + 2\sqrt{6}}R$, which is greater than R, and so $p_z(r) < 0$ on $[0, r_1]$. This indicates that to force the inner part of the plate to take the shape of a sphere, a force is required to push the plate up onto the sphere. This is clearly not a physically reasonable result, and so the plate cannot take the shape of a sphere on any finite region within $[0, R]$, including an annular region $[a, b]$, $0 < a < b \leq R$, due to contact alone. This indicates that perhaps a point load solution is more appropriate as a solution to this contact problem.

2.3 Point Source Solution

The solutions in (8) above are the deformations of a thin circular plate with a point load at its center, with simply supported and clamped edge conditions. We are interested in whether these solutions can be matched with the deformation due to contact with the sphere. If the sphere only touches the plate at $r = 0$, then for the simply supported and clamped boundary conditions we have that

$$P = \frac{16\pi D(R - c + h/2)(1 + \nu)}{r_0^2(3 + \nu)} \quad \text{and} \quad P = \frac{16\pi D(R - c + h/2)}{r_0^2}, \tag{12}$$

respectively. The deformations due to these point loads are then

$$\begin{aligned} w(r) &= \left(R - c + \frac{h}{2}\right)\left[1 - \rho^2 + \frac{1 + \nu}{3 + \nu}2\rho^2 \ln \rho\right] \\ \text{and} \qquad w(r) &= \left(R - c + \frac{h}{2}\right)\left(1 - \rho^2 + 2\rho^2 \ln \rho\right). \end{aligned} \tag{13}$$

Unfortunately, neither of these solutions are acceptable. For a finite length the surface of the sphere is below that of the plate, a physically impossible situation. We can see this by considering the radius of curvature of the solutions (13). Using the fact that the radius of curvature of the function $y = f(x)$ is $\sqrt{1 + (f'(x))^2} \Big/ f''(x)$ (with a negative solution indicating the center of the circle is down), derivatives of the solutions in (13) show that the curves have radius of curvature zero at $r = 0$. Since the sphere has radius of curvature R, near $r = 0$ the curve for the plate is above that of the sphere.

2.4 Ring Source Solution

Johnson [10] is a classic text on contact problems, and includes a two page section on plates and shells. The example considered in most detail is that of a flat plate of length $2l$, width w and thickness $2b$ deformed by contact with a rigid cylinder of radius R whose axis is perpendicular to the length of the plate, such that the contact arc is of length $2a$. Johnson then states that the contact loading is along two lines parallel to the axis of the cylinder at positions $x = \pm a$, and with a given load P the position of the contact a satisfies $P(l - a)/2 = 2Ewb^3/3R(1 - \nu^2)$. Without any more detail, Johnson observes that as the load increases, the pressure is concentrated at the edges. He also mentions, with reference to Essenburg [4], that the contact pressure due to a circle of contact is concentrated into a ring of force, which we will now investigate.

The deformations of a circular plate by a ring force at radius r_1 with force per unit length p and total load $P = 2\pi r_1 p$ with simple support and clamped edges are given by (9) and (10) respectively. After extensive algebra, one can show that these solutions are the same as using the approach that the plate is deformed a specific distance at $r = r_1$, with no other load, and continuity of deformation and its first derivative at $r = r_1$. In both cases we find that the second derivative of the deformation is continuous, and there is a discontinuity in the third derivative of magnitude $P/2\pi D r_1 = p/D$, the force per unit length of the ring force scaled by D. This result is encouraging, since we would expect a discontinuity in the third derivative of the solution of a fourth order equation with a point force.

Given that the deformation is caused by a sphere with center a distance c above the undeformed plate position, we wish to find the point r_1 and load P such that the ring source solution touches the sphere at $r = r_1$ with tangent the same as that of the sphere. For the simply supported case, we find from (9) and its derivative that we require

$$\left.\begin{aligned} \frac{P}{8\pi D}\left[2r_1^2 \ln\frac{r_1}{r_0} + \left(r_0^2 - r_1^2\right)\frac{(3+\nu)r_0^2 - (1-\nu)r_1^2}{2(1+\nu)r_0^2}\right] &= \frac{h}{2} - c + \sqrt{R^2 - r_1^2}, \\ \frac{P}{8\pi D}\left[2r_1 \ln\frac{r_1}{r_0} - (r_0^2 - r_1^2)\frac{(1-\nu)r_1}{(1+\nu)r_0^2}\right] &= \frac{-r_1}{\sqrt{R^2 - r_1^2}}. \end{aligned}\right\} \tag{14}$$

These equations can be rewritten as

$$P = \frac{8\pi D}{\sqrt{R^2 - r_1^2}} \bigg/ \left[\left(1 - \frac{r_1^2}{r_0^2}\right)\frac{(1-\nu)}{(1+\nu)} - 2\ln\frac{r_1}{r_0}\right], \tag{15}$$

and

$$\frac{\left[2r_1^2 \ln\frac{r_1}{r_0} + \left(r_0^2 - r_1^2\right)\frac{(3+\nu)r_0^2 - (1-\nu)r_1^2}{2(1+\nu)r_0^2}\right]}{\sqrt{R^2 - r_1^2}\left[\left(1 - \frac{r_1^2}{r_0^2}\right)\frac{(1-\nu)}{(1+\nu)} - 2\ln\frac{r_1}{r_0}\right]} = \frac{h}{2} - c + \sqrt{R^2 - r_1^2}. \tag{16}$$

For any given problem, one could specify r_0, ν, R, D, then given any c, solve (16) for the position of the ring load r_1, then substitute in (15) to explicitly find the total load. However, the approach we will take here is to choose various r_1, and see how c changes by solving (16) explicitly.

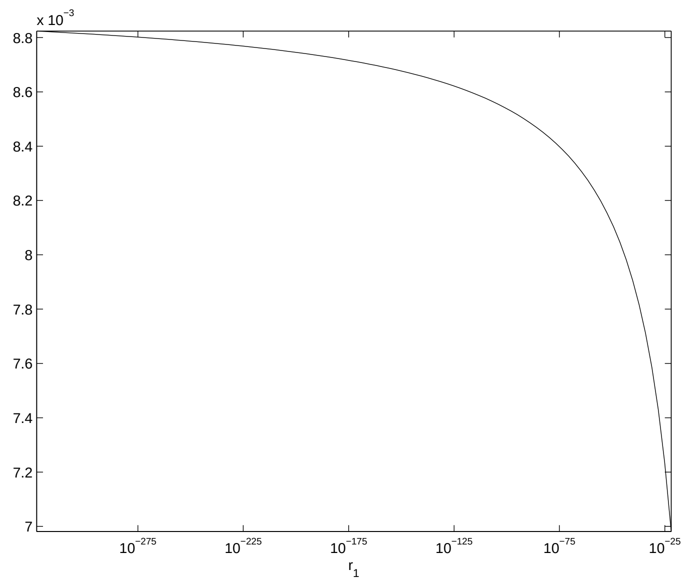

Figure 1: Sphere center c versus radius of ring load r_1.

For a particular case of interest [1], we choose $r_0 = 3.5 \times 10^{-2}$, $\nu = 0.4$, $R = 7.95 \times 10^{-3}$, $h = 2 \times 10^{-3}$ (lengths in meters), and are interested in varying r_1 so that c lies in the range of 8.95×10^{-3} down to 6.95×10^{-3}, which corresponds to a deformation of the plate of zero up to two millimeters. Figure 1 shows the results with these parameter values, where we found the limits of r_1 by trial and error. Noting in particular the logarithmic scale, we see that the position of the radius is at most 10^{-22} meters – about 9 orders of magnitude smaller than the nucleus of an atom. This is so incredibly small that it is beyond the limit at which the continuum hypothesis can be applied for an elastic solid. This tells us that despite the results of section 2.3 above, the contact between a rigid sphere and a thin plate, at least in the small deformation case, is to all intents and purposes a single point, so the loads and deformations in (12) and (13) respectively are the ones we should use.

3 Von Kármán Theory

3.1 Governing Equations

The work described in the previous section is considered valid if the deflection of the plate is small in comparison to the thickness of the plate. In the regime where linear elastic theory is still applicable, but the deformation is of the same order or larger than the plate thickness, the analysis needs to be adjusted. Lateral deflections of the middle surface of the plate are now accompanied by stretching, which can substantially increase the load-carrying capacity of the plate. The equations for large deformation of circular plates of constant thickness h with axisymmetric boundary conditions are (based on formulae in [2, 3]),

$$\begin{aligned} \nabla_r^2\nabla_r^2 w(r) &= \frac{h}{Dr}\left(\frac{d^2w}{dr^2}\frac{d\Phi}{dr} + \frac{dw}{dr}\frac{d^2\Phi}{dr^2}\right) + \frac{p_z(r)}{D}, \\ \nabla_r^2\nabla_r^2 \Phi(r) &= -\frac{E}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}, \end{aligned} \tag{17}$$

where Φ is an Airy type stress function. These equations are a special form of von Kármán's equations, first derived by him in 1910 [11]. An alternative formulation of these equations given in [2]) is,

$$\begin{aligned} \frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} &= \frac{12}{h^2}\frac{dw}{dr}\left(\frac{du}{dr} + \frac{\nu}{r}u + \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right) + \frac{1}{Dr}\int_0^r r p_z(r)dr, \\ \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u &= -\frac{(1-\nu)}{2r}\left(\frac{dw}{dr}\right)^2 - \frac{dw}{dr}\frac{d^2w}{dr^2}, \end{aligned} \tag{18}$$

where u is the radial displacement of the plate. We will use this formulation.

To solve the equations (18), we require a total of five boundary conditions, three for w as well as two for u. The requirement that w is an even function and u is odd (and so well behaved at $r = 0$) gives us two conditions $w'(0) = u(0) = 0$. For a circular plate simply supported at $r = r_0$, the two boundary conditions on w in (5) are imposed. The final boundary condition on u depends on whether the edge of the plate is allowed to move in the radial direction. If we allow the movement then Timoshenko and Woinowsky-Krieger [2] state the final boundary condition is $n_r(r_0) = 0$, that is there is no force in the radial direction on the outside edge of the plate. If we had formulated the equations in terms of a stress function Φ instead of u this condition is equivalent to $\Phi'(r_0) = 0$, which is given in the literature as a boundary condition for equations (17) (see for example Miersemann & Mittelman [12]). In terms of u and w this condition is

$$\nu\frac{u(r_0)}{r_0} + \frac{du}{dr}(r_0) + \frac{1}{2}\left(\frac{dw}{dr}(r_0)\right)^2 = 0. \tag{19}$$

3.2 Spherically Prescribed Deflection

If we assume that the plate is in contact with a sphere, as in section 2.2 above, then we have that $w(r) = h/2 - c + \sqrt{R^2 - r^2}$ for $0 \leq r \leq r_1$, where r_1 is the (initially unknown) radius of contact. We then need to solve the equations (18) in the regions $[0, r_1]$ and $[r_1, r_0]$. In $[0, r_1]$, w is given, and so we need to solve for u and p_z. In $[r_1, r_0]$, $p_z = 0$ and we need to solve for both w and u. Boundary conditions would be continuity of w and u and their derivatives across $r = r_1$, and satisfying these conditions should also give a position for r_1.

Unfortunately, an analytic solution to this problem is not available; the particular integral for u in $[0, r_1]$ cannot be evaluated in closed form. In any event, a solution in $[r_1, r_0]$ cannot be found analytically due to its nonlinear nature. A full numerical solution would be needed, which would also give the load p_z.

Solving equations (18) numerically for various load distributions using it was observed that the value of u is very small. For the particular case of interest with r_0, ν and R as given in section 2.4, $E = 2 \times 10^9$ and $h = 0.002$m, with a point load of magnitude $P = 66.3719$ resulting in a deflection of 2mm at the center, the maximum u is approximately 2.7×10^{-6}, and occurs at $r = 5.85$mm. To investigate whether contact over a region is possible we assume that within the contact region u is zero. As we expect the contact region, if it exists, to be small this assumption is reasonable. We can then use the first of the equations in (18) to solve for p_z. This gives

$$\int_0^r r p_z(r)\, dr = -D\frac{6r^2}{h^2}\left(R^2 - r^2\right)^{-5/2}\left(r^4 - \left(R^2 + \frac{h^2}{6}\right)r^2 + \frac{2h^2}{3}R^2\right). \tag{20}$$

Unfortunately equation (20) doesn't give us enough information about the load required for spherical deformation. To obtain the actual load function $p_z(r)$ instead of the integral we consider the fourth order equation:

$$\begin{aligned}&\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr}\\ &= \frac{p_z(r)}{D} + \frac{12}{h^2}\left(r\frac{dw}{dr}\left(\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right) + \frac{d^2w}{dr^2}\left(\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right)\right),\end{aligned} \tag{21}$$

obtained by differentiating the first equation in (18) with respect to r. Substituting $u = du/dr = 0$, $w = h/2 - c + \sqrt{R^2 - r^2}$ and derivatives and rearranging we find

$$p_z(r) = D(R^2 - r^2)^{-7/2}\left(\nu d r^6 + (1 - \nu d R^2 - 8dR^2)r^4 + 8R^2(dR^2 - 1)r^2 - 8R^4\right), \tag{22}$$

where $d = 6/h^2$. We can see that $p_z(0) = D(R^2)^{-7/2}(-8R^4) = -8D/R^3$. As for the small deflection case, since the load at $r = 0$ is negative and the z-axis is positive down, this implies that a force is pushing the plate upward onto the sphere. This rules out contact over a region including $r = 0$, however contact on an annular region $a \leq r \leq b$, where $0 < a \leq b < R$, may be possible, as $p_z(r)$ is not negative on the entire region $[0, R]$.

Note that the expression we have found for $p_z(r)$ does not depend on c, the position of the indenting sphere's center, but only on R, its radius. This is because the load we have found is just the force required for the plate to take the shape of the sphere, not the total force on the plate. The force required to shape the plate to the sphere depends only on the sphere's radius and the size of the region of contact, not on where the sphere is positioned.

3.3 Annular contact

From Gladwell [13], if there is contact on an annular region $a \leq r \leq b$ the load at the endpoints of this region should be zero, that is $p(a) = p(b) = 0$. We can use this to find the boundaries of the (possible) annular contact region for given constants. Continuing to work with the assumption that $u = 0$ over the contact region, the deflection of the plate w must satisfy the fourth order differential equation (21), with $p_z(r) = 0$ for $0 \leq r \leq a$, $b \leq r \leq r_0$. As the sphere is in contact with the plate over the region $a \leq r \leq b$, we know that in this region w is given by $w = h/2 - c + \sqrt{R^2 - r^2}$. The edges of the contact region, a and b, are the zeros of the load function (22).

With $R = 0.00795$ the contact region (a, b) is $a = 0.00082557$ and $b = 0.00786583$, regardless of the value of c. The total load on the plate, calculated by numerically integrating $rp_z(r)$ over (a, b) (using Matlab's quadl routine), comes out as $P \approx 5660000$. Again this is independent of c. This load seems excessively large.

Figure 2 shows a plot of the solution, for $c = 0.007$, over $(0, b)$ with the points $(-r_0, 0)$ and $(r_0, 0)$, the simply supported boundary of the plate, marked with asterisks. The dotted line shows the middle surface of the plate, the solid line touching the sphere is the upper surface of the plate. It can be seen that the plate is in contact with the sphere over such a large region that parts of the plate are above zero. The solution over (b, r_0) can not be calculated as the numerical method used to find the solution on $[0, b)$ will not converge in this situation. Figure 3 shows the small deflection analytic solution for the middle surface of the plate for the same values of R, c, a and b, with the borders of the contact region marked by the asterisks.

It is clear that this solution is not realistic. The contact between the sphere and the plate cannot extend past the point where $w = 0$. For $w(b) \leq 0$, the position of the sphere center c must be less than 0.001154. However this would result in a deflection at $r = 0$ greater than 0.0068m, which is obviously larger than the maximum permissable deflection of 2mm. We suspect that the extremely large value of P calculated corresponds to the situation where the sphere center is positioned in such a way that annular contact over (a, b) gives a reasonable shape to the deflected plate, if not a reasonable scale. For

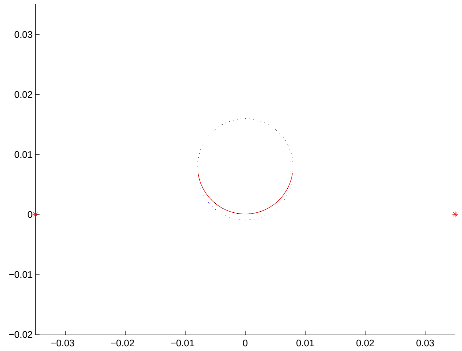

Figure 2: Large deflection solution for $R = 0.00795, c = 0.005$, annular contact over (a, b)

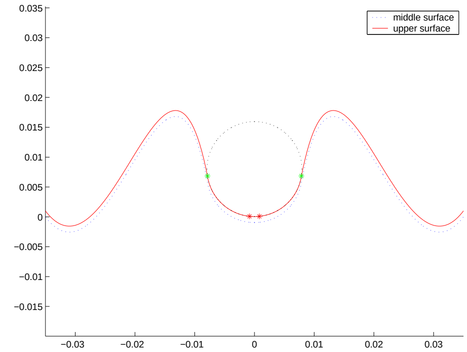

Figure 3: Small deflection solution for annular contact over (a, b)

this to occur c would have to be large and negative. This would result in a deflection much larger than the maximum allowable deflection of 2mm.

3.4 Contact on a ring

We have determined that contact over a region is not possible. We now consider contact on a ring. In this analysis we no longer use the simplifying assumption that $u = 0$. However we assume that the thickness of the plate after it is deformed is still uniform and given by h. In reality this is not the case due to the stretching of the plate but since u is small the change in the thickness of the plate as a result of this stretching is negligible. If the contact between the sphere of radius R and the plate occurs along a ring of radius r_1, then we have the equation

$$R^2 = (w(r_1) - h/2 + c)^2 + (u(r_1) + r_1)^2, \tag{23}$$

where c is the position of the sphere center. Rearranging equation (23) we find

$$c = h/2 - w(r_1) + \sqrt{R^2 - (u(r_1) + r_1)^2}. \tag{24}$$

Figure 4 is a plot of c against r_1 for different load values, using $R = 0.00795$. The values of c in Figure 4 are reasonable, implying that contact on a ring is possible. However this choice of R is arbitrary. Figure 5 is a plot of the deformed shape of the sphere for $r_1 = 0.0025$ with a sphere of this radius and we can see that while the equation gives a reasonable value of c and the deformed plate does touch the sphere at r_1, the sphere passes through the plate. This is obviously not physically possible.

If we also impose the condition that the derivative of the deformation must be continuous at r_1, that is the slope of the plate must be smooth, then implicitly differentiating the

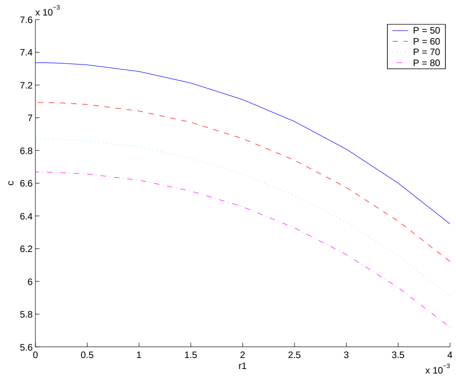

Figure 4: position of sphere center c against r_1

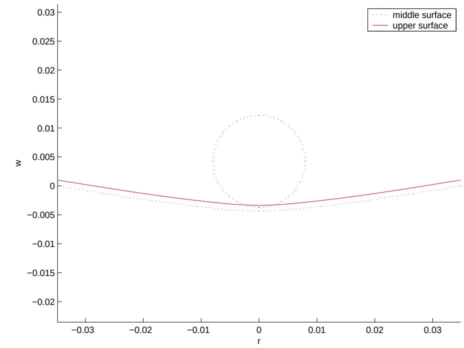

Figure 5: plate and indenting sphere $P = 250, r_1 = 0.0025, R = 0.00795$

equation

$$c = h/2 - w(r) + \sqrt{R^2 - (u(r) + r)^2}$$

gives

$$\frac{dw}{dr} = -(R^2 - (u(r) + r)^2)^{-1/2}(u + r)\left(\frac{du}{dr} + 1\right). \tag{25}$$

We can use this to calculate the value of R. Rearranging equation (25) and evaluating it at $r = r_1$ gives

$$R = \left((u(r_1) + r_1)^2 \left(1 + \left(\frac{u'(r_1) + 1}{w'(r_1)}\right)^2\right)\right)^{1/2}. \tag{26}$$

For $r_1 = 0.0025$ and $P = 250$, the values used to produce Figure 5, we find $R \approx 0.05$. This is larger than the radius of the plate $r_0 = 0.035$. Figure 6 shows the plate with a sphere of this radius. The sphere is never below the plate, and there is contact at r_1, however the physical situation we are interested in involves the indentation of a plate by a much smaller sphere.

Figures 7 and 8 are plots of R against r_1 for different load values. Figure 7 also shows (as dotted lines which appear slightly below the large deflection plots for the same load values) R calculated using $u = u' = 0$ and the small deflection solution (9) to find $w'(r_1)$. Figure 8 has an asterisk at the point corresponding to the values of P, R and r_1 for Figure 6.

It is obvious that for small loads R is larger than the radius of the plate. However as P increases, R becomes smaller. Perhaps if the load is large enough, a ring load may become possible. Figure 9 is a plot of R against the load magnitude P for $r_1 = 0.00005$. With a point load at the center (which results in a larger deflection at $r = 0$ than the

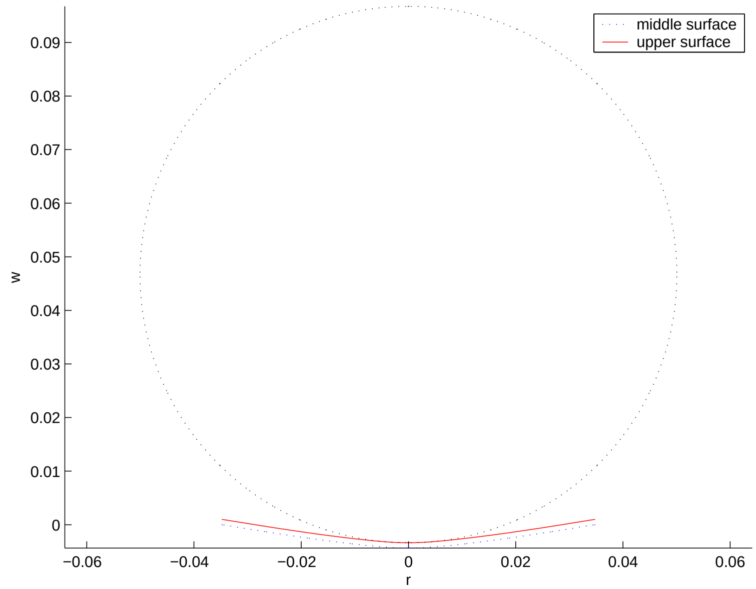

Figure 6: plate and sphere $P = 250$, $r_1 = 0.0025$, R calculated from (26)

ring load of the same magnitude) a deflection of 2mm or 2×10^{-3}m, the largest allowable, is achieved with a load $P \approx 66.3719$. A reasonable value of R is 0.00795. With $P = 300$, R is still larger than this ($R \approx 0.0126$), but we cannot have a load this large and achieve reasonable values for the deflection. To have maximum deflection ($w(0)$) in the range 0 to 2mm requires c in the range $R + h/2 - 0.002$ to $R + h/2$, that is 0.00695 to 0.00895. Figure 10 shows the indenting sphere's radius R and center position c as r_1 varies for $P = 66.3719$. For $r = 1 \times 10^{-5}$, we have $c \approx 0.0382$ and $R \approx 0.0392$, which are nowhere near our required values.

R also decreases as r_1 decreases. With tiny r_1 it is expected that R would become reasonable. We know that this is the case for small loads. However it is impossible to solve numerically for r_1 small enough to see this.

3.5 Point contact

The radius of curvature of a function $f(x)$ is given by $r_c = \left(\sqrt{1 + (f'(x))^2}\right) / (f''(x))$. The radius of curvature of the sphere is obviously of magnitude R. The radius of curvature of the plate at $r = 0$, where $u = 0$, will be

$$r_c = \frac{\sqrt{1 + (w'(0))^2}}{w''(0)} = \frac{1}{w''(0)},$$

since $w'(0) = 0$.

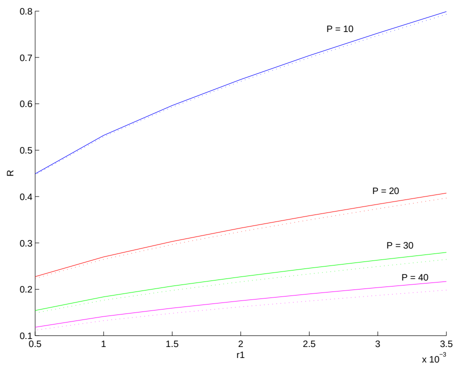

Figure 7: radius of indenting sphere R against r_1, $P = 10, 20, 30, 40$

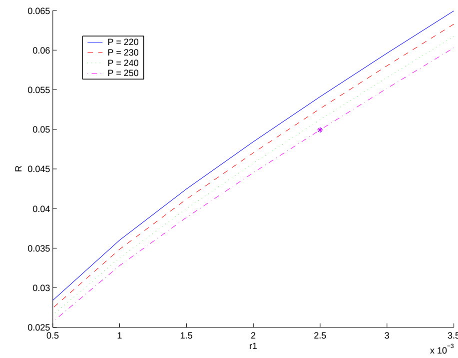

Figure 8: radius of indenting sphere R against r_1, $P = 220, 230, 240, 250$

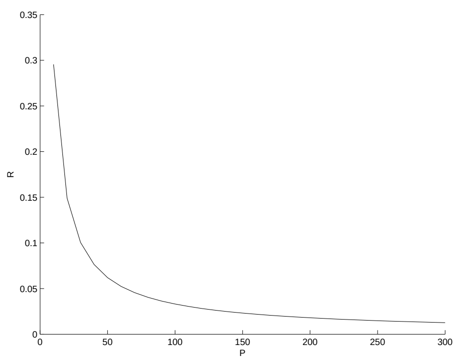

Figure 9: radius of indenting sphere R against P, $r_1 = 0.00005$

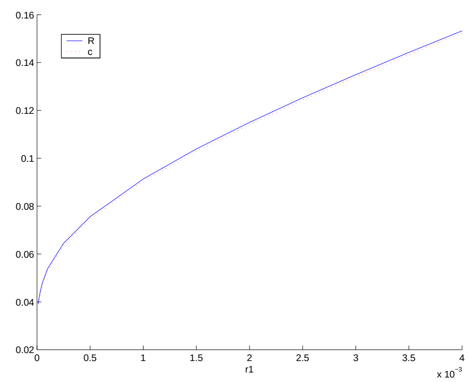

Figure 10: R and c against $r1$

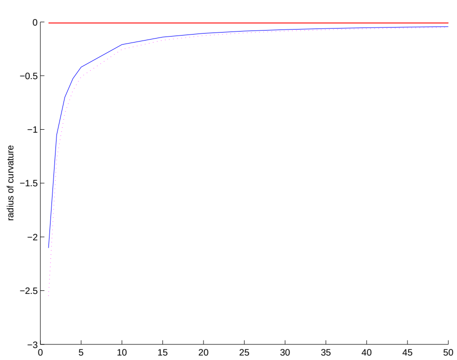

Figure 11: radius of curvature against P, $n = 2800$

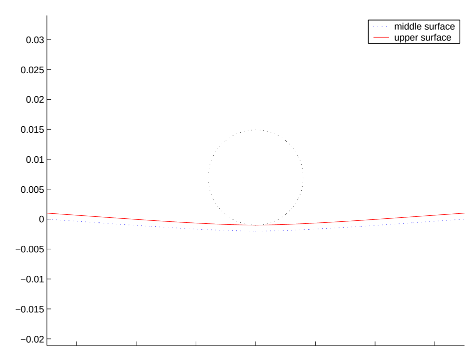

Figure 12: plate and indenting sphere for point load $P = 66.3719$, $R = 0.00795$

The solid line in Figure 11 is a plot of the radius of curvature of the plate under various loads, for our chosen h, v and E. A point solution is physically impossible if the magnitude of the radius of curvature of the plate is smaller than that of the sphere as this would imply that the sphere is below the plate for some finite length near $r = 0$. The straight line across the plot near the top is $R = 0.00795$ so this appears to show that a point load is possible.

It was stated in section 2.3 above that the radius of curvature for the plate using small deflection theory is zero at $r = 0$. This implies that for some finite length near $r = 0$, the surface of the sphere is below that of the plate, a physical impossibility, so the sphere cannot touch the plate at just a single point. The dotted line in Figure 11 is the radius of curvature calculated using small deflection with $r = 0.0003$. This seems to show that the radius of curvature at $r = 0$ is indeed zero in the larger deflection case also and it is just numerical error which is showing point contact to be possible. When the number of points used to evaluate the numerical solutions from which the radius of curvature was calculated is increased the estimate of the radius of curvature decreases in magnitude.

We may find the radius of curvature for large deflection analytically by using a series expansion solution for w and u around $r = 0$. As the load term contains $1/r$ we will need to have a negative power in the series expansions. Let w, which is even, $(w'(0) = w^{(3)}(0) = 0)$, be given by

$$w = a_{-2}r^{-2} + a_0 + a_2r^2 + a_4r^4 + a_6r^6 + \dots$$

and u, which is odd, $(u(0) = u''(0) = 0)$, by

$$u = b_{-1}r^{-1} + b_1r + b_3r^3 + b_5r^5 + \dots.$$

Substituting these expressions and their derivatives into the first of equations (18) with a point load at the center so that the load term $1/Dr \int_0^r rp_z(r)dr = P/2\pi Dr$, we find the radius of curvature to be

$$r_c = \frac{1}{w''(0)} = \frac{1}{2a_2} = \frac{24(1-\nu)b_{-1}\pi D}{Ph^2}.$$

We know that $u(0) = 0$, and we have approximated u by $u = b_{-1}r^{-1} + b_1r + b_3r^3 + \dots$ so it is possible for $u(0) = 0$ only if $b_{-1} = 0$. This means the radius of curvature at $r = 0$ is zero.

For the small deflection case, despite the radius of curvature result, the conclusion drawn was that for small deflection contact between the sphere and plate is effectively at a point as the contact occurs over a ring of tiny radius. This result still holds for large deflection theory for small loads (resulting in deformation within the region of applicability of the small deflection theory) as the solutions are identical. Figure 12 shows the deformed plate with the indenting sphere for a load $P = 66.3719$, which produces a maximum deflection of 2mm, the thickness of the plate. The plot does not provide any evidence that point contact between the sphere and plate is not possible. For $P = 10, r = 1e{-}109$, the radius of curvature from the small deflection analytic expression is -0.0079565. So the radius of curvature of the plate is of magnitude less than R only for $r < 1e-109$. This means that the sphere is definitely above the plate for $r \geq 1e-109$, so the finite length near $r = 0$, for which the surface of the sphere is below that of the plate, is so tiny that we will never see it on a plot.

4 Conclusions

The analysis has shown that for both large and small deflection theory the contact between the sphere and the plate is practically at a point. It was demonstrated analytically for small deflection theory that the contact is over a ring of such small magnitude that it is in practical terms at a point. For large deflection this will also be the case. It has been shown that contact over a region, either annular or including $r = 0$, is not possible so the contact must be either over a ring or at a point. It appears that contact over a ring would be possible if it were numerically possible to solve for small enough r_1 so the contact is most likely again over a ring of tiny magnitude. For large deflection contact at a point is possible numerically if not analytically, so in both numerical and practical terms the contact between the sphere and plate is at a point.

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