# Integral proofs that $355/113 > \pi$

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## 1 Introduction

One of the more beautiful results related to approximating $\pi$ is the integral

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi. \tag{1}$$

Since the integrand is nonnegative on the interval $[0, 1]$, this shows that $\pi$ is strictly less than 22/7, the well known approximation to $\pi$. The first published statement of this result was in 1971 by Dalzell [3], although anecdotal evidence [2] suggests it was known by Kurt Mahler in the mid-1960s. The result (1) is not hard to prove, if perhaps somewhat tedious. A partial fraction decomposition leads to a polynomial plus a term involving $1/(1 + x^2)$, which integrates immediately to the required result. An alternative is to use the substitution $x = \tan\theta$, leading to a polynomial in powers of $\tan\theta$. We then apply the recurrence relation for taking the integrals of powers of $\tan\theta$. Of course, the simplest approach today is to simply verify (1) using a symbolic manipulation package such as Maple or Mathematica.

An obvious question at this point might be whether similar elegant integral results can be found for other rational approximations for $\pi$. A particularly good approximation is 355/113, which is accurate to seven digits. Our aim here is to find a variety of such integral results.

## 2 More general integrals

Backhouse [1] recently generalized (1) by varying powers in the numerator of the integrand, leading to

$$I_{m,n} = \int_0^1 \frac{x^m(1-x)^n}{1+x^2}\,dx = a + b\pi + c\ln(2), \tag{2}$$

where $a, b, c$ are rationals that depend on the positive integers $m$ and $n$, and $a$ and $b$ have opposite sign. Backhouse [1] showed that if $2m - n \equiv 0 \pmod 4$, then $c = 0$ and a variety of approximation to $\pi$ are obtained. An integral equal to $a + b\pi$ leads to a rational approximation of $\pi$ as $|a/b|$, and the maximum value of the integrand gives an upper bound on the error. As $m$ and $n$ increase,

| $m$ | $n$ | $I_{m,n}$ | $I$ error | $J_{m,n}$ | $J$ error |
|---|---|---|---|---|---|
| 1 | 2 | $\pi/2 - 3/2$ | $1.4e-1$ | $5/3 - \pi/2$ | $1.9e-1$ |
| 3 | 2 | $19/12 - \pi/2$ | $2.5e-2$ | $6/5 - 3\pi/8$ | $5.8e-2$ |
| 2 | 4 | $\pi - 47/15$ | $8.3e-3$ | $49\pi/32 - 24/5$ | $6.9e-3$ |
| 1 | 6 | $63/10 - 2\pi$ | $8.4e-3$ | $681/35 - 99\pi/16$ | $3.0e-3$ |
| 5 | 2 | $\pi/2 - 47/30$ | $8.3e-3$ | $104/105 - 5\pi/16$ | $2.8e-2$ |
| 4 | 4 | $22/7 - \pi$ | $1.3e-3$ | $323\pi/256 - 416/105$ | $1.5e-3$ |
| 3 | 6 | $2\pi - 1759/280$ | $5.2e-4$ | $5018/315 - 649\pi/128$ | $2.6e-4$ |
| 7 | 2 | $1321/840 - \pi/2$ | $3.6e-3$ | $272/315 - 35\pi/128$ | $1.6e-2$ |
| 2 | 8 | $3959/315 - 4\pi$ | $4.7e-4$ | $10439\pi/512 - 20176/315$ | $1.0e-4$ |
| 6 | 4 | $\pi - 1979/630$ | $3.2e-4$ | $563\pi/512 - 1088/315$ | $5.1e-4$ |
| 1 | 10 | $8\pi - 15829/630$ | $9.2e-4$ | $178555/693 - 20995\pi/256$ | $9.4e-5$ |
| 5 | 6 | $377/60 - 2\pi$ | $7.4e-5$ | $48008/3465 - 1129\pi/256$ | $4.7e-5$ |
| 9 | 2 | $\pi/2 - 989/630$ | $1.9e-3$ | $128/165 - 63\pi/256$ | $1.1e-2$ |
| 4 | 8 | $4\pi - 4838/385$ | $3.4e-5$ | $36231\pi/2048 - 64192/1155$ | $9.5e-6$ |
| 8 | 4 | $10886/3465 - \pi$ | $1.1e-4$ | $2023\pi/2048 - 512/165$ | $2.2e-4$ |

Table 1. Comparing $I_{m,n}$ and $J_{m,n}$

the integrand becomes increasingly flat (Backhouse calls them "pancake functions") and the approximation to $\pi$ improved. Unfortunately, no form of (2) leads to the approximation 355/113.

Another question raised in [1, 2] is whether there are other families of integrals with positive integrands that can be used to give increasingly accurate estimates of $\pi$. One such class of integrals not previously considered is

$$J_{m,n} = \int_0^1 \frac{x^m(1-x)^n}{\sqrt{1-x^2}}\, dx = a + b\pi, \tag{3}$$

where $a$ and $b$ are rationals of opposite sign depending on the positive integers $m$ and $n$. Note that an approximation to $\pi$ can be found from (3) for every choice of positive $m$ and $n$. For comparison purposes, table 1 lists $I_{m,n}$ and $J_{m,n}$ for the first few $m$ and $n$ for which $I_{m,n}$ does not involve $\ln(2)$, as well as the error in the approximation to $\pi$. While the errors are similar, in no case is the approximation 355/113 forthcoming.

## 3 Some integral results

In what follows, we shall look at a variety of ways of forming integrals with positive integrands that evaluate to $355/113 - \pi$.

### 3.1 **Idea 1 – set** $b = 1$

The first idea is to multiply the integrands in (2) and (3) by a positive rational $\alpha$ so that the coefficient multiplying $\pi$ becomes one, and the integral equals $a' - \pi$. Note that this limits our choices of $m, n$ to when $b$ is positive. We then require $m, n$ large enough that $a' < 355/113$. If we add $\beta = 355/113 - a'$ to both sides, we get

$$\int_0^1 \left[\frac{\alpha x^m(1-x)^n}{1+x^2} + \beta\right] dx = \frac{355}{113} - \pi, \tag{4}$$

and

$$\int_0^1 \left[\frac{\alpha x^m(1-x)^n}{\sqrt{1-x^2}} + \beta\right] dx = \frac{355}{113} - \pi, \tag{5}$$

where $\alpha$ and $\beta$ (different for the two different integrals, of course) are positive rationals. Since the integrands in (4,5) are nonnegative, this gives the required integral proof that $355/113 > \pi$. Numerical testing with the criteria of choosing $\beta$ with the smallest number of digits leads to the best results

$$\int_0^1 \left[\frac{x^{10}(1-x)^8}{4(1+x^2)} + \frac{5}{138450312}\right] dx = \frac{355}{113} - \pi, \tag{6}$$

and

$$\int_0^1 \left[\frac{8192}{114291}\frac{x^9(1-x)^8}{\sqrt{1-x^2}} + \frac{15409}{219772564011}\right] dx = \frac{355}{113} - \pi. \tag{7}$$

Now while these integrals do indeed have positive integrands, in a sense they are not really giving us an approximation involving 355/113. We have simply chosen $m$ and $n$ large enough so that the approximations to $\pi$ given by $I_{m,n}$ and $J_{m,n}$ are better than 355/113, then degraded the results to get the approximation 355/113 by adding a “fudge factor” to the integrand. While strictly correct, they lack a certain elegance.

### 3.2 **Idea 2 – set** $a = 355/113$

Another approach is to multiply the integrands of (2) or (3) by a positive rational $\alpha$ so that the integral equals $355/113 - b'\pi$. If we choose $m$ and $n$ large enough that $b'$ is a positive rational greater than one, then we can choose $\beta = 4(b' - 1)$ and $\beta' = 2(b' - 1)$ so that

$$\int_0^1 \frac{\alpha x^m(1-x)^n + \beta}{1+x^2}\, dx = \frac{355}{113} - \pi \quad \text{and} \quad \int_0^1 \frac{\alpha x^m(1-x)^n + \beta'}{\sqrt{1-x^2}}\, dx = \frac{355}{113} - \pi. \tag{8}$$

These results use the integrals $\int_0^1 1/(1+x^2)\, dx = \pi/4$ and $\int_0^1 1/\sqrt{1-x^2}\, dx = \pi/2$, and since the integrands are nonnegative, once again lead to $355/113 > \pi$.

Numerical testing with various values of $m$ and $n$ lead to the simplest results in terms of the number of digits in $\alpha$ and $\beta$ as

$$\int_0^1 \frac{21747726x^{10}(1-x)^8+4}{86990903(1+x^2)}\,dx = \frac{355}{113} - \pi, \tag{9}$$

and

$$\int_0^1 \left[\frac{355355}{10535216}x^m(1-x)^n + \frac{15409}{345217957888}\right] \Big/ \sqrt{1-x^2}\,dx = \frac{355}{113} - \pi. \tag{10}$$

Unfortunately, the previous argument can also be used against the elegance of these results, its just the position of the addition to the integral to degrade the approximation that has changed.

### 3.3 Idea 3 – multiply by a polynomial

Finally, we can take the approach of multiplying the integrand by a low order polynomial, and adjusting the coefficients to return the correct result. We then choose the simplest case where the polynomial is nonnegative on $[0,1]$ – simplest in the sense of the smallest number of digits in the coefficients. Experimenting with $I_{m,n}$ leads to the results

$$\int_0^1 \frac{x^7(1-x)^7(192-791x+983x^2)}{3164(1+x^2)}\,dx = \frac{355}{113} - \pi, \tag{11}$$

and

$$\int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{3164(1+x^2)}\,dx = \frac{355}{113} - \pi. \tag{12}$$

While the powers are lower in (11), the number of characters in the expression is larger than in (12), and it is not immediately obvious that the integrand is nonnegative. The integral (12) is nearly as simple as (1), and it is immediately obvious that the integrand is nonnegative. The best case for $J_{m,n}$ is

$$\int_0^1 \frac{x^7(1-x)^8(617273-478592x)}{16288272\sqrt{1-x^2}}\,dx = \frac{355}{113} - \pi, \tag{13}$$

which only uses a linear factor. The results using quadratic factors are more complicated.

## 4 Conclusion

After testing a variety of families of integrals (some of which are not mentioned here), the result (12) is the simplest formula available that represents the gap between 355/113 and $\pi$ as the integral of a positive integrand. Since $1 < 1+x^2 < 2$ for the integral range, we can use it to form the bound

$$\frac{355}{113} - \frac{911}{2630555928} < \pi < \frac{335}{113} - \frac{911}{5261111856}, \tag{14}$$

or $355/113 - 3.46 \times 10^{-7} < \pi < 335/113 - 1.73 \times 10^{-7}$. The true result is $\pi = 355/113 - 2.67 \times 10^{-7}$, which is very close to the middle of the bound.

## References

[1] N. Backhouse, *Note 79.36, Pancake functions and approximations to* $\pi$, Math. Gazette **79** (1995), 371–374.

[2] J.M. Borwein, *The life of Pi, history and computation*, seminar presentation 2003, available from `http://www.cecm.sfu.ca/personal/jborwein/pi-slides.pdf`, March 2005.

[3] D.P. Dalzell, *On* 22/7 *and* 355/113, Eureka: the Archimedian's Journal **34** (1971), 10–13.

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