So You Think You Can Divide?

A History of Division

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Tobias Dantzig: Number (1930, p26)

"There is a story of a German merchant of the fifteenth century, which I have not succeeded in authenticating, but it is so characteristic of the situation then existing that I cannot resist the temptation of telling it. It appears that the merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which in his opinion was the only country where such advanced instruction could be obtained."

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- Ancient Techniques
  - Definitions
  - Successive Subtraction
  - Doubling
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- Positional Notation
  - Positional Definition
  - Galley or Scratch
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  - Napier's Rods and the "Modern" method
  - Short Division and Genaille's Rods
  - Double Division
- Division Yielding Decimals
  - Integer Division
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  - Multiply by Reciprocal
  - Iteration – Newton
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  - Iteration – EDSAC

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Definitions

If a and b are natural numbers and $a = qb + r$, where q is a nonnegative integer and r is an integer satisfying $0 \leq r < b$, then q is the quotient and r is the remainder after integer division. Also, a is the dividend and b is the divisor.

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Or, $100 = 0 \cdot 12 + 100 = 1 \cdot 12 + 88 = 2 \cdot 12 + 76 = 3 \cdot 12 + 64 = 4 \cdot 12 + 52 = 5 \cdot 12 + 40 = 6 \cdot 12 + 28 = 7 \cdot 12 + 16 = 8 \cdot 12 + 4$.

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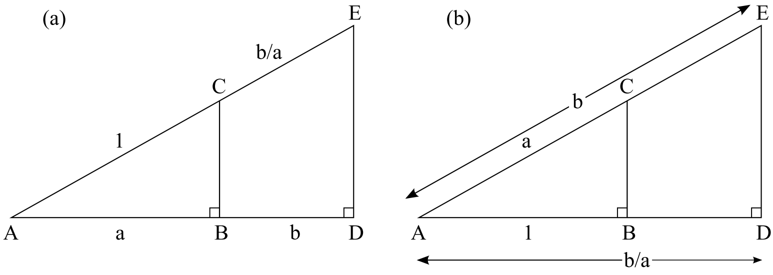
$1 + 2 + 4 + 64 = 71$,
so $1652/23 = 71$ r 19.

Geometry



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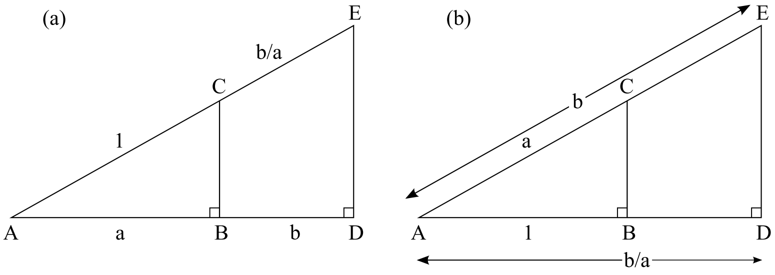

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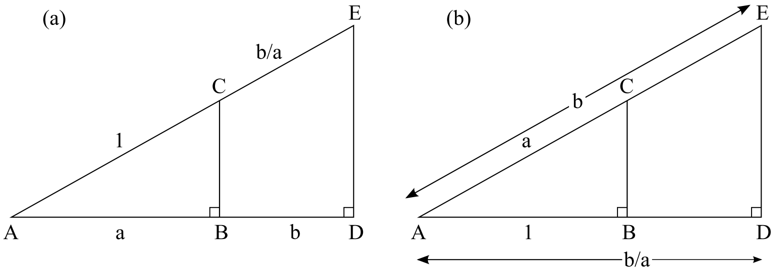

(a): $a < 1$ so $a/1 = (a+b)/(1+CE)$, $a + aCE = a + b$, $CE = b/a$.

(b): $a > 1$ so $a/1 = b/AD$, $AD = b/a$.

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It would be nice to set out the computation more cleanly...

Galley or Scratch Division

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e.g. 9592/47: $47 \times 1 = 47$, $47 \times 2 = 94$, $47 \times 3 = 141$, $47 \times 4 = 188$, $47 \times 5 = 235$, $47 \times 6 = 282$, $47 \times 7 = 329$, $47 \times 8 = 376$, $47 \times 9 = 423$.

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$$\begin{array}{cccc|ccc} & \not 6 & \not 5 & 4 & & & \\ \not 4 & \not 8 & \not 8 & \not 3 & 6 & 9 & 7 \\ \not 7 & \not 7 & \not 7 & \not 7 & & & \end{array}$$

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$$\begin{array}{cccc|ccc} \not 1 & 1 & & & & & \\ \not 9 & \not 5 & 9 & 2 & 2 & 0 & 4 \\ \not 4 & \not 7 & \not 7 & 7 & & & \\ & \not 4 & 4 & & & & \end{array}$$

Galley or Scratch Division

Originally developed by the Hindus, most popular method in Europe until the end of the 17th century. Successively subtract multiples of the divisor appropriately shifted.

e.g. 4883/7:

$$\begin{array}{cccc|ccc} & \cancel{6} & \cancel{5} & 4 & & & \\ \cancel{4} & \cancel{8} & \cancel{8} & \cancel{3} & 6 & 9 & 7 \\ \cancel{7} & \cancel{7} & \cancel{7} & \cancel{7} & & & \end{array}$$

e.g. 9592/47: $47 \times 1 = 47$, $47 \times 2 = 94$, $47 \times 3 = 141$, $47 \times 4 = 188$, $47 \times 5 = 235$, $47 \times 6 = 282$, $47 \times 7 = 329$, $47 \times 8 = 376$, $47 \times 9 = 423$.

$$\begin{array}{cccc|ccc} \cancel{1} & \cancel{1} & 3 & & & & \\ \cancel{9} & \cancel{5} & \cancel{9} & 2 & 2 & 0 & 4 \\ \cancel{4} & \cancel{7} & \cancel{7} & 7 & & & \\ & \cancel{4} & \cancel{4} & & & & \end{array}$$

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Originally developed by the Hindus, most popular method in Europe until the end of the 17th century. Successively subtract multiples of the divisor appropriately shifted.

e.g. 4883/7:

	~~6~~	~~5~~	4				
~~4~~	~~8~~	~~8~~	~~3~~	|	6	9	7
~~7~~	~~7~~	~~7~~	~~7~~				

e.g. 9592/47: $47 \times 1 = 47$, $47 \times 2 = 94$, $47 \times 3 = 141$, $47 \times 4 = 188$, $47 \times 5 = 235$, $47 \times 6 = 282$, $47 \times 7 = 329$, $47 \times 8 = 376$, $47 \times 9 = 423$.

~~1~~	~~1~~	~~3~~	4				
~~9~~	~~5~~	~~9~~	~~2~~	|	2	0	4
~~4~~	~~7~~	~~7~~	~~7~~				
	~~4~~	~~4~~					

Factor Division

Fibonacci suggested splitting the divisor if possible: $a \div (bc) = (a \div b) \div c$. Galley division is easier when the divisor is small, particularly single digits.

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$24\,286 = 8095 \times 3 + 1$, $8095 = 1156 \times 7 + 3$, $1156 = 144 \times 8 + 4$.

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Working backwards, $8095 = (144 \times 8 + 4) \times 7 + 3 = 144 \times 8 \times 7$
$+4 \times 7 + 3 = 144 \times 56 + 31$,

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$24\,286 = 8095 \times 3 + 1$, $8095 = 1156 \times 7 + 3$, $1156 = 144 \times 8 + 4$.

Working backwards, $8095 = (144 \times 8 + 4) \times 7 + 3 = 144 \times 8 \times 7$ $+4 \times 7 + 3 = 144 \times 56 + 31$, $24\,286 = (144 + 31) \times 3 + 1$ $= 144 \times 56 \times 3 + 31 \times 31 + 1 = 144 \times 168 + 94$.

Factor Division

Fibonacci suggested splitting the divisor if possible: $a \div (bc) = (a \div b) \div c$. Galley division is easier when the divisor is small, particularly single digits.

E.g. $24\,286 \div 168 = 24\,286 \div (3 \times 7 \times 8)$.

$24\,286 = 8095 \times 3 + 1$, $8095 = 1156 \times 7 + 3$, $1156 = 144 \times 8 + 4$.

Working backwards, $8095 = (144 \times 8 + 4) \times 7 + 3 = 144 \times 8 \times 7 + 4 \times 7 + 3 = 144 \times 56 + 31$, $\quad 24\,286 = (144 + 31) \times 3 + 1 = 144 \times 56 \times 3 + 31 \times 31 + 1 = 144 \times 168 + 94$.

But, finding factors means more division problems, factors may not be small, and multiple single digit divisions aren't easier than a single division.

Napier's Rods

0	1	2	3	4	5	6	7	8	9
0/0	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8
0/0	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7
0/0	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
0/0	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
0/0	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
0/0	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
0/0	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
0/0	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

Napier's Rods, Divisor Multiples, the Modern Method

1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
0/4	3/2	2/8	3/2	3512
0/5	4/0	3/5	4/0	4390
0/6	4/8	4/2	4/8	5268
0/7	5/6	4/9	5/6	6146
0/8	6/4	5/6	6/4	7024
0/9	7/2	6/3	7/2	7902

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1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
0/4	3/2	2/8	3/2	3512
0/5	4/0	3/5	4/0	4390
0/6	4/8	4/2	4/8	5268
0/7	5/6	4/9	5/6	6146
0/8	6/4	5/6	6/4	7024
0/9	7/2	6/3	7/2	7902

244 392/878:

$$8\ 7\ 8\,\overline{)\ 2\ 4\ 4\ 3\ 9\ 2}$$

Napier's Rods, Divisor Multiples, the Modern Method

1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
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0/6	4/8	4/2	4/8	5268
0/7	5/6	4/9	5/6	6146
0/8	6/4	5/6	6/4	7024
0/9	7/2	6/3	7/2	7902

244 392/878:

							2		
8	7	8)	2	4	4	3	9	2	
			1	7	5	6			

Napier's Rods, Divisor Multiples, the Modern Method

1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
0/4	3/2	2/8	3/2	3512
0/5	4/0	3/5	4/0	4390
0/6	4/8	4/2	4/8	5268
0/7	5/6	4/9	5/6	6146
0/8	6/4	5/6	6/4	7024
0/9	7/2	6/3	7/2	7902

244 392/878:

$$
\begin{array}{r|cccccc}
 & & & & 2 & & \\
\hline
8\ 7\ 8\ ) & 2 & 4 & 4 & 3 & 9 & 2 \\
 & 1 & 7 & 5 & 6 & & \\
\cline{2-5}
 & & 6 & 8 & 7 & 9 &
\end{array}
$$

Napier's Rods, Divisor Multiples, the Modern Method

1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
0/4	3/2	2/8	3/2	3512
0/5	4/0	3/5	4/0	4390
0/6	4/8	4/2	4/8	5268
0/7	5/6	4/9	5/6	6146
0/8	6/4	5/6	6/4	7024
0/9	7/2	6/3	7/2	7902

244 392/878:

```
                    2   7
          -------------------------
8  7  8 ) 2   4   4   3   9   2
          1   7   5   6
          ---------------
              6   8   7   9
              6   1   4   6
```

Napier's Rods, Divisor Multiples, the Modern Method

1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
0/4	3/2	2/8	3/2	3512
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0/8	6/4	5/6	6/4	7024
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244 392/878:

```
                2  7
        ________________
8 7 8 ) 2  4  4  3  9  2
        1  7  5  6
        __________
           6  8  7  9
           6  1  4  6
           __________
              7  3  3  2
```

Napier's Rods, Divisor Multiples, the Modern Method

1	8	7	8	
0/2	1/6	1/4	1/6	1756
0/3	2/4	2/1	2/4	2634
0/4	3/2	2/8	3/2	3512
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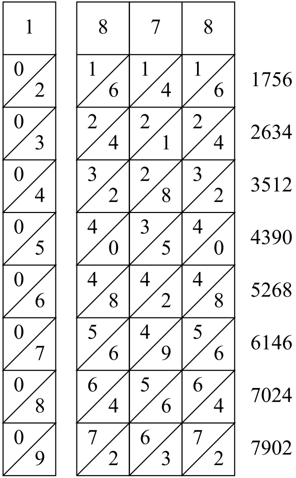
0/8	6/4	5/6	6/4	7024
0/9	7/2	6/3	7/2	7902

244 392/878:

```
                        2   7   8
            ---------------------
8  7  8 )   2   4   4   3   9   2
            1   7   5   6
            ----------------
                6   8   7   9
                6   1   4   6
                ----------------
                    7   3   3   2
                    7   0   2   4
```

Napier's Rods, Divisor Multiples, the Modern Method



244 392/878:

```
                2 7 8
      ___________________
8 7 8 ) 2 4 4 3 9 2
        1 7 5 6
        -------
          6 8 7 9
          6 1 4 6
          -------
            7 3 3 2
            7 0 2 4
            -------
              3 0 8
```

Other Layouts

English speaking world, China, Japan, India:

$$\begin{array}{r} 6\,9\,7 \\ 7\,\overline{)\,4\,8\,8\,3} \\ \underline{4\,2} \\ 6\,8 \\ \underline{6\,3} \\ 5\,3 \\ \underline{4\,9} \\ 4 \end{array}$$

Other Layouts

English speaking world, China, Japan, India:

$$\begin{array}{r} 697 \\ 7\,\overline{)\,4883} \\ \underline{42} \\ 68 \\ \underline{63} \\ 53 \\ \underline{49} \\ 4 \end{array}$$

Much of Latin America:

$$\begin{array}{l} 4883 \div 7 = 697 \\ \underline{42} \\ 68 \\ \underline{63} \\ 53 \\ \underline{49} \\ 4 \end{array}$$

Other Layouts

English speaking world, China, Japan, India:

```
      6 9 7
   --------
7 ) 4 8 8 3
    4 2
    ---
      6 8
      6 3
      ---
        5 3
        4 9
        ---
          4
```

Much of Latin America:

```
4 8 8 3 ÷ 7 = 697
4 2
---
  6 8
  6 3
  ---
    5 3
    4 9
    ---
      4
```

Mexico:

```
      6 9 7
   --------
7 ) 4 8 8 3
      6 8
        5 3
          4
```

More Layouts

Spain, Italy, France, Portugal, Romania, Russia:

$$
\begin{array}{rcccc|l}
 & 4 & 8 & 8 & 3 & 7 \\
\cline{6-6}
- & 4 & 2 & & & 697 \\
\cline{2-3}
 & & 6 & 8 & & \\
 & - & 6 & 3 & & \\
\cline{3-4}
 & & & 5 & 3 & \\
 & & - & 4 & 9 & \\
\cline{4-5}
 & & & & 4 &
\end{array}
$$

Brazil and Colombia, no | before quotient.

More Layouts

Spain, Italy, France, Portugal, Romania, Russia:

```
   4 8 8 3|7
 - 4 2    |697
   6 8
 - 6 3
     5 3
   - 4 9
       4
```

Brazil and Colombia, no | before quotient.

France:

```
   4 8 8 3|7
 - 4 2    |697
     6 8  |
   - 6 3  |
       5 3|
     - 4 9|
         4|
```

No decimals.

More Layouts

Spain, Italy, France, Portugal, Romania, Russia:

$$
\begin{array}{rrrr|l}
 4 & 8 & 8 & 3 & 7 \\ \cline{5-5}
-4 & 2 & & & 697 \\ \cline{1-2}
 & 6 & 8 & & \\
 - & 6 & 3 & & \\ \cline{2-3}
 & & 5 & 3 & \\
 & - & 4 & 9 & \\ \cline{3-4}
 & & & 4 &
\end{array}
$$

Brazil and Colombia, no | before quotient.

France:

$$
\begin{array}{rrrr|l}
 4 & 8 & 8 & 3 & 7 \\ \cline{5-5}
-4 & 2 & & & 697 \\ \cline{1-2}
 & 6 & 8 & & \\
 - & 6 & 3 & & \\ \cline{2-3}
 & & 5 & 3 & \\
 & - & 4 & 9 & \\ \cline{3-4}
 & & & 4 &
\end{array}
$$

No decimals.

Germany, Norway, Poland, Croatia, Slovenia, Hungary, Czech Republic, Slovakia, Bulgaria:

$$
\begin{array}{rrrrl}
 4 & 8 & 8 & 3 & : 7 = 697 \\
-4 & 2 & & & \\ \cline{1-2}
 & 6 & 8 & & \\
 - & 6 & 3 & & \\ \cline{2-3}
 & & 5 & 3 & \\
 & - & 4 & 9 & \\ \cline{3-4}
 & & & 4 &
\end{array}
$$

Short Division

With single digit divisors, each step will involve at most two digit numbers, and subtraction leaves a one digit number.

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With single digit divisors, each step will involve at most two digit numbers, and subtraction leaves a one digit number.

		6	9	7
7)	4	8	8	3
	4	2		
		6	8	
		6	3	
			5	3
			4	9
				4

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Short Division

With single digit divisors, each step will involve at most two digit numbers, and subtraction leaves a one digit number. For compactness, carry the single digit to the left as a subscript – short division, as opposed to traditional long division.

$$
\begin{array}{r|cccc}
 & & 6 & 9 & 7 \\ \cline{2-5}
7\,) & 4 & 8 & 8 & 3 \\
 & 4 & 2 & & \\ \cline{2-3}
 & & 6 & 8 & \\
 & & 6 & 3 & \\ \cline{3-4}
 & & & 5 & 3 \\
 & & & 4 & 9 \\ \cline{4-5}
 & & & & 4
\end{array}
$$

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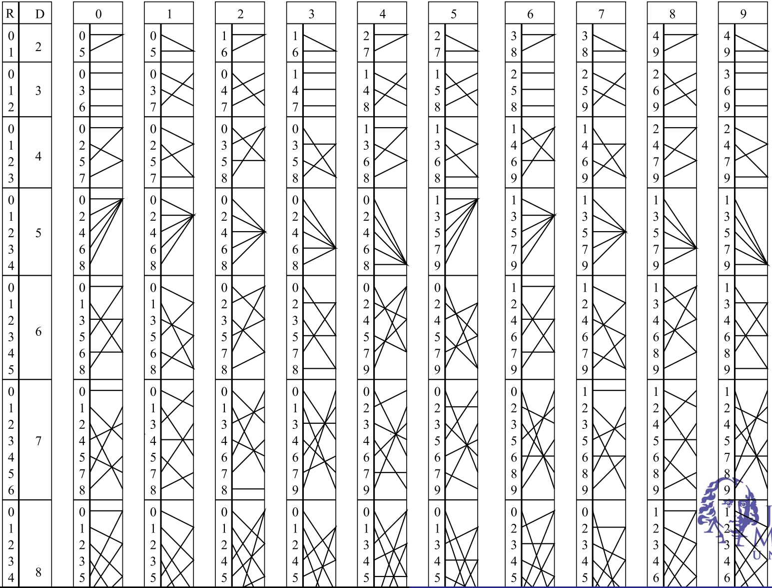
With single digit divisors, each step will involve at most two digit numbers, and subtraction leaves a one digit number. For compactness, carry the single digit to the left as a subscript – short division, as opposed to traditional long division.

$$\begin{array}{r@{\;}cccc} & & 6 & 9 & 7 \\ \cline{2-5} 7\,) & 4 & 8 & 8 & 3 \\ & 4 & 2 & & \\ \cline{2-3} & & 6 & 8 & \\ & & 6 & 3 & \\ \cline{3-4} & & & 5 & 3 \\ & & & 4 & 9 \\ \cline{4-5} & & & & 4 \end{array}$$

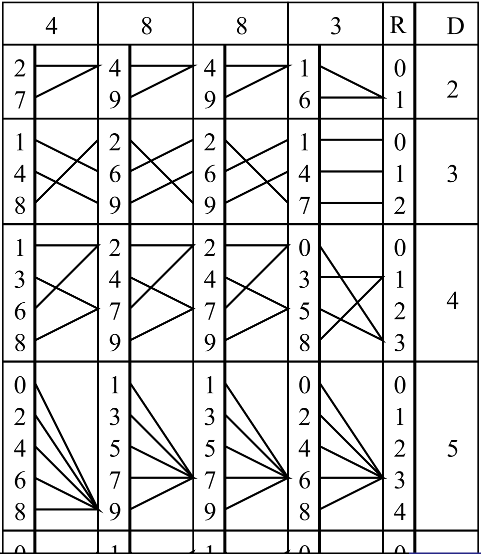
or

$$\begin{array}{r@{\;}ccccc} & & 6 & 9 & 7 & \\ \cline{2-6} 7\,) & 4 & 8 & {}^{6}8 & {}^{5}3 & {}^{4} \end{array}$$

Genaille's Rods for Short Division



Example: 4883÷?

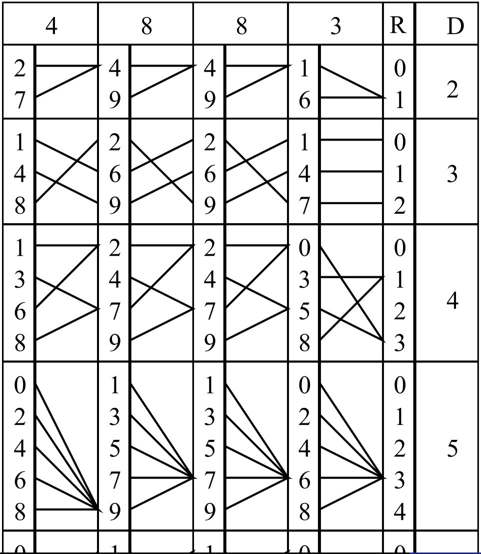


Example: 4883÷?

4	8	8	3	R	D
2 7	4 9	4 9	1 6	0 1	2
1 4 8	2 6 9	2 6 9	1 4 7	0 1 2	3
1 3 6 8	2 4 7 9	2 4 7 9	0 3 5 8	0 1 2 3	4
0 2 4 6 8	1 3 5 7 9	1 3 5 7 9	0 2 4 6 8	0 1 2 3 4	5
0	1	1	0	0	

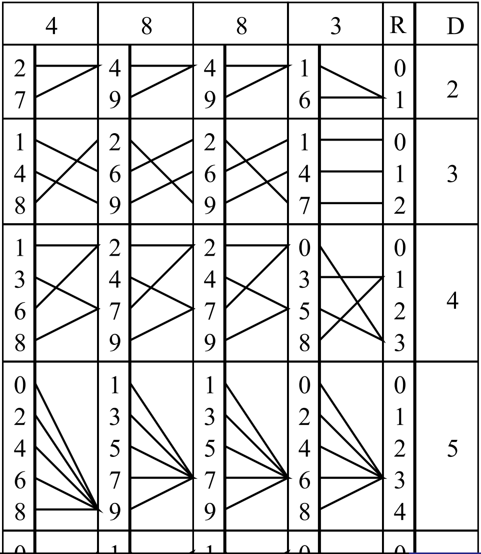
So
$4883 \div 2 = 2441$ r 1,

Example: 4883÷?



So
$4883 \div 2 = 2441$ r 1,
$4883 \div 3 = 1627$ r 2,

Example: 4883÷?



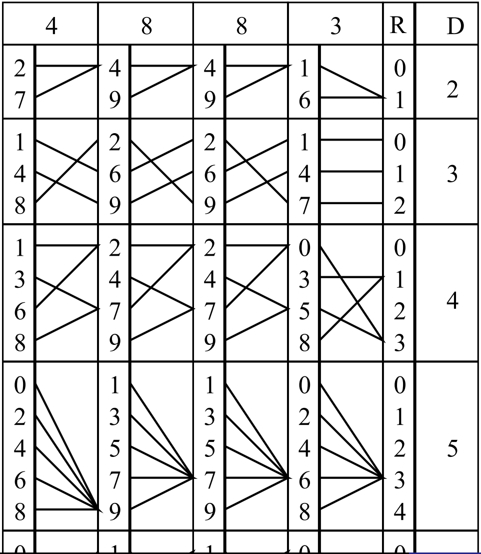
So

$4883 \div 2 = 2441$ r 1,

$4883 \div 3 = 1627$ r 2,

$4883 \div 4 = 1220$ r 3,

Example: 4883÷?



So
$4883 \div 2 = 2441$ r 1,
$4883 \div 3 = 1627$ r 2,
$4883 \div 4 = 1220$ r 3,
$4883 \div 5 = 976$ r 3,
and so on.

Double Division

Chunking (UK, late 1990's) takes away "easy" (100, 10, 5, 2 etc.) multiples of the divisor.

Double Division

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Double Division

$$7 \;)\; \overline{4 \quad 8 \quad 8 \quad 3}$$

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Double Division

$1_\times$	7	$\overline{)\ 4\ 8\ 8\ 3}$
$2_\times$	14	
$4_\times$	28	
$8_\times$	56	

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Double Division

$1\times$	7	)	4	8	8	3		
$2\times$	14		2	8	0	0		400
$4\times$	28		2	0	8	3		
$8\times$	56							

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Double Division

$1_\times$	7	)	4	8	8	3	
$2_\times$	14		2	8	0	0	400
$4_\times$	28		2	0	8	3	
$8_\times$	56		1	4	0	0	200
				6	8	3	

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Double Division

1×	7	)	4	8	8	3		
2×	14		2	8	0	0		400
4×	28		2	0	8	3		
8×	56		1	4	0	0		200
				6	8	3		
				5	6	0		80
				1	2	3		

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Double Division

1×	7	)	4	8	8	3		
2×	14		2	8	0	0		400
4×	28		2	0	8	3		
8×	56		1	4	0	0		200
				6	8	3		
				5	6	0		80
				1	2	3		
					7	0		10
					5	3		

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Double Division

$1_\times$	7	)	4	8	8	3	
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$4_\times$	28		2	0	8	3	
$8_\times$	56		1	4	0	0	200
				6	8	3	
				5	6	0	80
				1	2	3	
					7	0	10
					5	3	
					2	8	4
					2	5	

Chunking (UK, late 1990's) takes away "easy" (100, 10, 5, 2 etc.) multiples of the divisor. Double division (Jeff Wilson, 2005) uses the first three doubles of the divisor as the chunks.

Double Division

$1\times$	7	)	4	8	8	3	
$2\times$	14		2	8	0	0	400
$4\times$	28		2	0	8	3	
$8\times$	56		1	4	0	0	200
				6	8	3	
				5	6	0	80
				1	2	3	
					7	0	10
					5	3	
					2	8	4
					2	5	
					1	4	2
					1	1	

Chunking (UK, late 1990's) takes away "easy" (100, 10, 5, 2 etc.) multiples of the divisor. Double division (Jeff Wilson, 2005) uses the first three doubles of the divisor as the chunks.

Double Division

1×	7	) 4	8	8	3		
2×	14	2	8	0	0	400	
4×	28	2	0	8	3		
8×	56	1	4	0	0	200	
			6	8	3		
			5	6	0	80	
			1	2	3		
				7	0	10	
				5	3		
				2	8	4	
				2	5		
				1	4	2	
				1	1		
					7	1	
					4		

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Double Division

$1_\times$	7	)	4	8	8	3	
$2_\times$	14		2	8	0	0	400
$4_\times$	28		2	0	8	3	
$8_\times$	56		1	4	0	0	200
				6	8	3	
				5	6	0	80
				1	2	3	
					7	0	10
					5	3	
					2	8	4
					2	5	
					1	4	2
					1	1	
						7	1
						4	697

Chunking (UK, late 1990's) takes away "easy" (100, 10, 5, 2 etc.) multiples of the divisor. Double division (Jeff Wilson, 2005) uses the first three doubles of the divisor as the chunks.

Another Example

$$214 \;\;) \overline{\;7 \quad 3 \quad 4 \quad 8 \quad 5\;}$$

Another Example

$$
\begin{array}{lr}
1_\times & 214 \;\big)\; \overline{7 \quad 3 \quad 4 \quad 8 \quad 5} \\
2_\times & 428 \\
4_\times & 856 \\
8_\times & 1712
\end{array}
$$

Another Example

$$
\begin{array}{lr}
1_\times & 214 \\
2_\times & 428 \\
4_\times & 856 \\
8_\times & 1712
\end{array}
\qquad
\begin{array}{r|ccccc|l}
214 & \overline{7} & \overline{3} & \overline{4} & \overline{8} & \overline{5} & \\
 & 4 & 2 & 8 & 0 & 0 & 200 \\
\hline
 & 3 & 0 & 6 & 8 & 5 &
\end{array}
$$

Another Example

$1_\times$	214	)	7	3	4	8	5		
$2_\times$	428		4	2	8	0	0		200
$4_\times$	856		3	0	6	8	5		
$8_\times$	1712		2	1	4	0	0		100
				9	2	8	5		

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				9	2	8	5		
				8	5	6	0		40
					7	2	5		

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$8_\times$	1712		2	1	4	0	0	100
				9	2	8	5	
				8	5	6	0	40
					7	2	5	
					4	2	8	2
					2	9	7	

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Another Example

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$4_\times$	856		3	0	6	8	5	
$8_\times$	1712		2	1	4	0	0	100
				9	2	8	5	
				8	5	6	0	40
					7	2	5	
					4	2	8	2
					2	9	7	
					2	1	4	1
						8	3	

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4×	856		3	0	6	8	5	
8×	1712		2	1	4	0	0	100
				9	2	8	5	
				8	5	6	0	40
					7	2	5	
					4	2	8	2
					2	9	7	
					2	1	4	1
						8	3	343

Using Integer Division

Given that ($p < q$) $\dfrac{p}{q} = \dfrac{a_{-1}}{10} + \dfrac{a_{-2}}{10^2} + \dfrac{a_{-3}}{10^3} + \cdots$, successively multiply by ten, stop if repeated or zero remainder.

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For example, $\frac{11}{16} = \frac{a_{-1}}{10} + \frac{a_{-2}}{10^2} + \frac{a_{-3}}{10^3} + \cdots$.

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so $a_{-1} = 6$ and $\frac{7}{8} = \frac{a_{-2}}{10} + \frac{a_{-3}}{10^2} + \frac{a_{-4}}{10^3} + \cdots$.

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so $a_{-2} = 8$ and $\frac{3}{4} = \frac{a_{-3}}{10} + \frac{a_{-4}}{10^2} + \frac{a_{-5}}{10^3} + \cdots$.

Integer Division Example, continued

Times 10: $10 \times \frac{3}{4} = \frac{30}{4} = 7\frac{1}{2}$

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$$\text{Times 10: } 10 \times \frac{1}{2} = 5$$

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Of course, process could also be periodic.

Decimal Long Division

Each of the steps of the previous example (11/16) is equivalent to one step in dividing 110 000 by 16, just shifted to deal with the position of the decimal. So we can use standard long division with a decimal point.

Decimal Long Division

$$
\begin{array}{r|ccccccccccc}
 & & & & 2 & 2 & . & 3 & 1 & 6 & 1 & \\
\cline{2-11}
990 & 2 & 2 & 0 & 9 & 3 & . & 0 & 0 & 0 & 0 & \cdots \\
 & 1 & 9 & 8 & 0 & & & & & & & \\
\cline{2-5}
 & & 2 & 2 & 9 & 3 & . & & & & & \\
 & & 1 & 9 & 8 & 0 & . & & & & & \\
\cline{3-6}
 & & & 3 & 1 & 3 & . & 0 & & & & \\
 & & & 2 & 9 & 7 & . & 0 & & & & \\
\cline{4-8}
 & & & & 1 & 6 & . & 0 & 0 & & & \\
 & & & & & 9 & . & 9 & 0 & & & \\
\cline{5-9}
 & & & & & 6 & . & 1 & 0 & 0 & & \\
 & & & & & 5 & . & 9 & 4 & 0 & & \\
\cline{6-10}
 & & & & & & & 1 & 6 & 0 & 0 & \\
 & & & & & & & & 9 & 9 & 0 & \\
\cline{8-11}
 & & & & & & & & & 6 & 1 & \\
 & & & & & & & & & & & \ddots
\end{array}
$$

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For example,
$22093/990 =$
$22.3161616161\ldots.$

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This technique was used by the Ancient Babylonians in base sixty – which is a highly composite number, so most reciprocals have a finite radix sixty representation.

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For example $1/7$ with $x_0 = 0.2$,
$x_1 = x_0(2 - 7 \times x_0) = 0.2(2 - 7 \times 0.2) = 0.12$,
$x_2 = x_1(2 - 7 \times x_1) = 0.12(2 - 7 \times 0.12) = 0.1392$,
$x_3 = x_2(2 - 7 \times x_2) = 0.1392(2 - 7 \times 0.1392) = 0.14276352$,
$x_4 = x_3(2 - 7 \times x_3) = 0.14276352(2 - 7 \times 0.14276352) = 0.1428570815004672$. True is 0.142857142857143.

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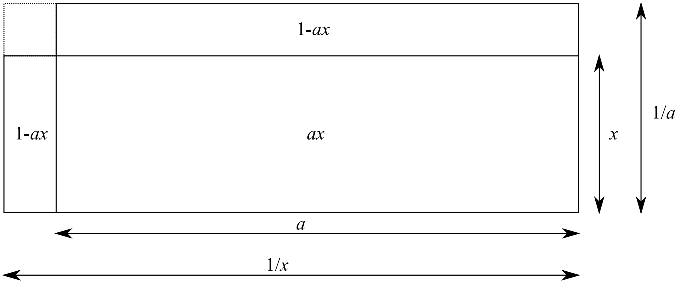
0.1428570815004672. True is 0.142857142857143.

Accuracy doubles at every step, given a close enough initial guess.

Graphical Proof



If $x \approx 1/a$ and we ignore the top left rectangle, then $(1/x) \cdot (1/a) \approx ax + 2(1 - ax)$, $1/a \approx x(ax + 2 - 2ax)$, or $1/a \approx x(2 - ax)$.

Goldschmidt's Iteration

Robert Goldschmidt, 1964 M.I.T. Masters dissertation.
To calculate p/q, start with y_0 close to $1/q$. Then
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To move the denominator closer to one, let $q_1 = 1 + e_1$. Then let $p_2/q_2 = (p_1 y_1)/(q_1 y_1)$ where $y_1 = 1 - e_1$. So $p_2 = p_1(1 - e_1)$ and $q_2 = (1 + e_1)(1 - e_1) = 1 - e_1^2$. Eliminating e_1 we can write $y_1 = 2 - q_1$.

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Algorithm: given p_i and q_i where $q_i \approx 1$, let $y_i = 2 - q_i$ then $p_{i+1} = y_i p_i$ and $q_{i+1} = y_i q_i$.

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Algorithm: given p_i and q_i where $q_i \approx 1$, let $y_i = 2 - q_i$ then $p_{i+1} = y_i p_i$ and $q_{i+1} = y_i q_i$.

Identical to Newton with $p_i = x_{i-1}$ and $q_i = ax_{i-1}$. So why use it? The two multiplications can be done in parallel, essentially doubling the speed!

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Let $c_1 = 1 - qy_0$, so $\frac{p}{q} = \frac{p_1}{1-c_1} \cdot \frac{1+c_1}{1+c_1} = \frac{p_1(1+c_1)}{1-c_1^2} = \frac{p_2}{1-c_2}$ where $p_2 = p_1(1+c_1)$ and $c_2 = c_1^2$.

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Let $c_1 = 1 - qy_0$, so $\dfrac{p}{q} = \dfrac{p_1}{1 - c_1} \cdot \dfrac{1 + c_1}{1 + c_1} = \dfrac{p_1(1 + c_1)}{1 - c_1^2} = \dfrac{p_2}{1 - c_2}$ where $p_2 = p_1(1 + c_1)$ and $c_2 = c_1^2$.

In general, $p_{i+1} = (1 + c_i)p_i$ and $c_{i+1} = c_i^2$ with $c_1 = 1 - qy_0$.

Conclusion

So, just how would you like to divide now?