Different Differential Equations with the Same Solution

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Special Session 39

Outline

- What happens: different differential equations, different solutions
- Why it happens: error analysis
- Relations to the Power Series Method
- Symplectic solvers
- Effectively symplectic solver
- Future Work

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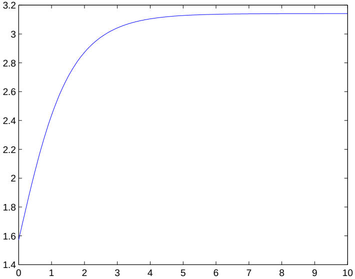

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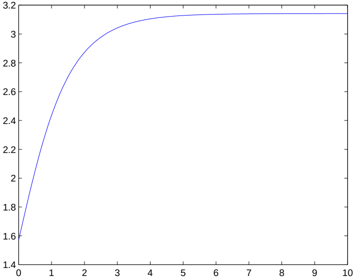

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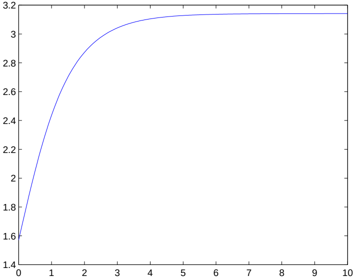

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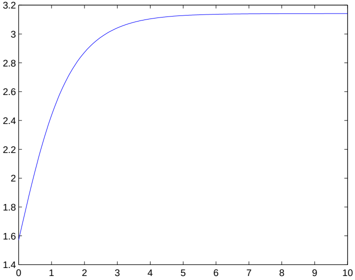

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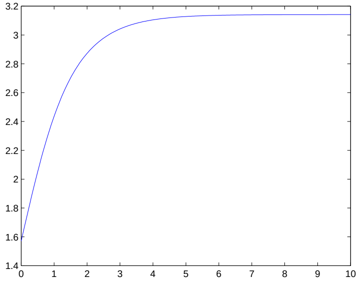

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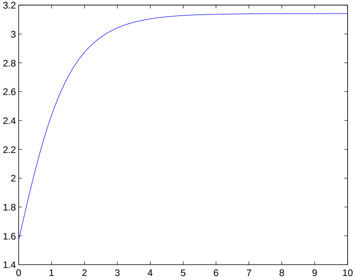

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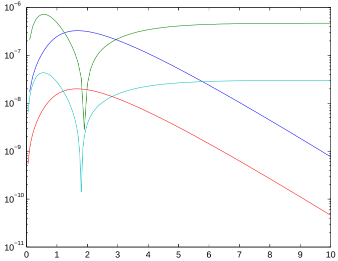
Trig Errors

Equal step RKO4 on $[0, 10]$ with 100, 200 intervals:



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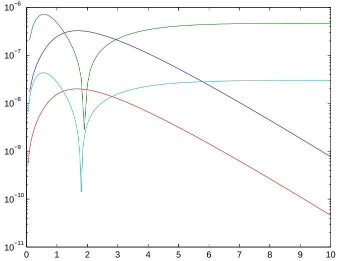
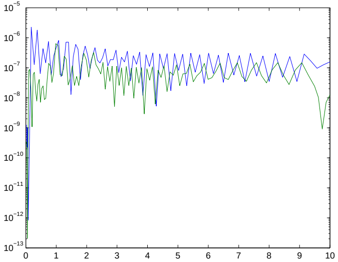


Matlab's ode45 on $[0, 10]$, absolute error 10^{-6}: 85 and 109 function evaluations.



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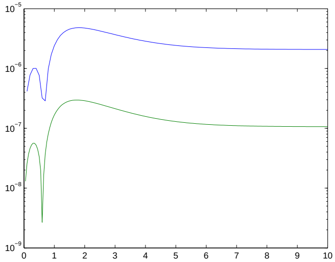

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Equal step on $[0, 10]$ with 100, 100 intervals, order 4, 6, 8, 10:

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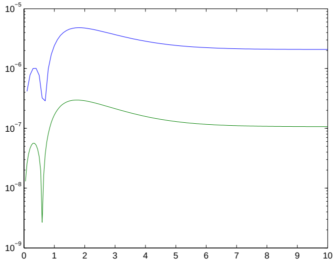

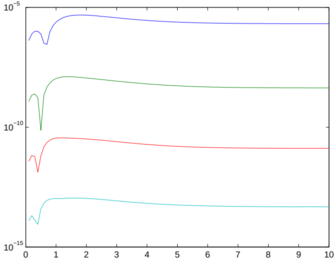

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Or more simply $u_1' = u_1 u_4$, $u_4' = (\alpha-1) u_4^2$.

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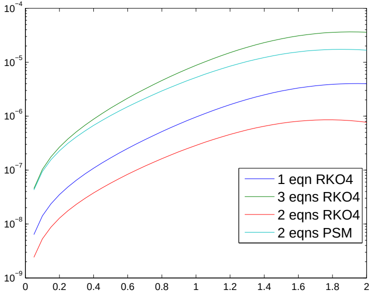

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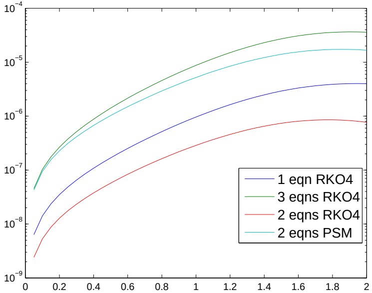

5th order PSM slightly better than RKO4 with two equations, twelfth order gives machine accuracy (or 100 intervals at eighth order).

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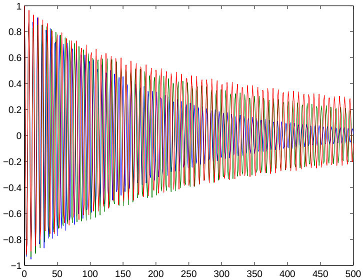
Letting $u_1 = \theta$, $u_2 = \theta'$, $u_3 = \sin\theta$, $u_4 = \cos\theta$, then $u_1' = u_2$, $u_2' = -u_3$, $u_3' = u_2 u_4$, $u_4' = -u_2 u_3$ with $u_1(0) = \theta_0$, $u_2(0) = 0$, $u_3(0) = \sin(\theta_0)$, $u_4(0) = \cos(\theta_0)$.

Pendulum Errors

500 intervals on $[0, 500]$ angle



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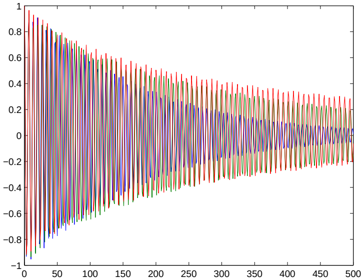

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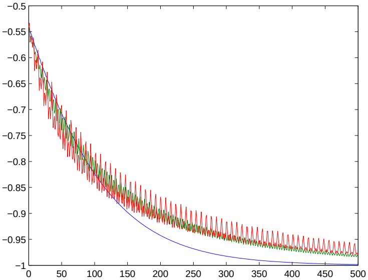

Pendulum Errors 2

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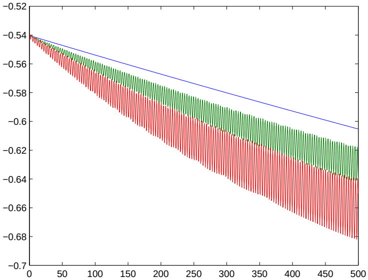

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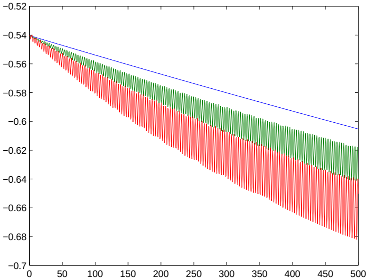

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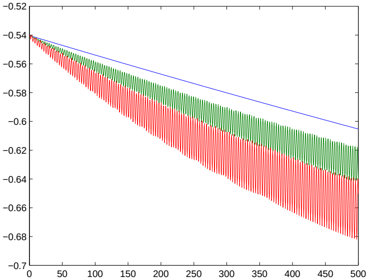

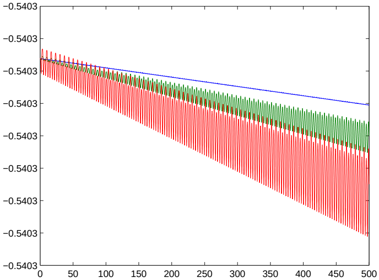

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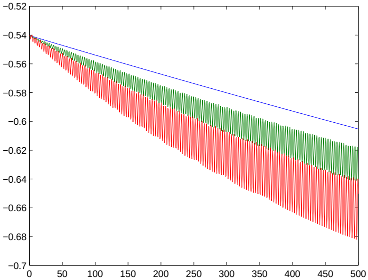

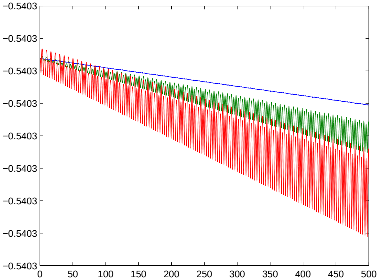

10 000 intervals on $[0, 500]$ $\Delta e = 2.9 \times 10^{-6}$. PSM with order 8 has $\Delta e = 2.2 \times 10^{-13}$, and order 12 energy is constant to machine precision.

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- Power Series Method order 4 solutions usually have slightly more error than Runge-Kutta order 4 solutions for the same polynomial systems – and usually require more computational work!
- But, the PSM can be made of arbitrary order, and has many other advantages...

Error Analysis, Initial Idea

Euler's method for $y' = f(t, y(t))$ is $y_{n+1} = y_n + h\,f(t_n, y_n)$ with local error $O(h^2)$ and global error $O(h)$.

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Higher order derivations almost always stop here, and leave the impression that error is proportional to h and depends on derivatives of y. But (for Euler), since $y''(\xi) = \frac{\partial f}{\partial t}(\xi, y(\xi)) + \frac{\partial f}{\partial y}(\xi, y(\xi))f(\xi, y(\xi))$, we can relate the error to the RHS.

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Error in u_1 is $\dfrac{h}{2}(\sin(\xi_1)\cos(\xi_1)) + O(h^2)$.

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- Polynomial form is superior because the rhs is much more efficient to evaluate – compare a few of multiplications to evaluating transcendental functions.
- Every system of odes can be rewritten in polynomial form in an algorithmic manner. Functions like $(e^t - 1)/t$ and $(\sin t)/t$ at $t = 0$ can cause problems.

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Computationally, each multiplication in Runge-Kutta requires Cauchy products in PSM:

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i.$$

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  - Makes delay differential equations trivial to solve.
- Arbitrary order means we can easily push to machine accuracy solutions, regardless of the required precision. For Hamiltonian systems of odes, we can conserve energy to machine precision – effectively symplectic.

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Forest-Ruth (1990) is fourth order: Given x_i, v_i: $x_a = x_i + \theta h v_i/2$, $v_a = v_i + \theta h f(x_a)$, $x_b = x_a + (1-\theta)hv_a/2$, $v_b = v_a + (1-2\theta)hf(x_b)$, $x_c = x_b + (1-\theta)hv_b/2$, $v_{i+1} = v_b + \theta h f(x_c)$, $x_{i+1} = x_c + \theta h v_{i+1}/2$ with $\theta = 1/(2-\sqrt[3]{2}) \approx 1.35$.

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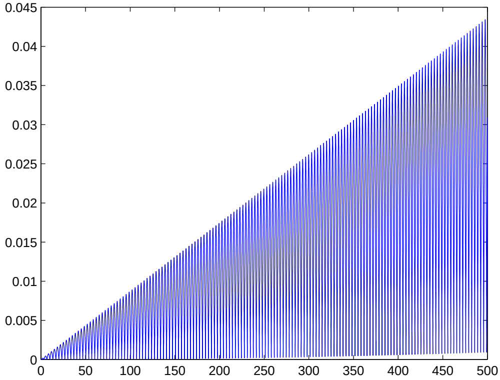

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- Symplectic solvers for more complicated Hamiltonians can be very complicated.
- (Apparently,) all symplectic algorithms require equal step sizes.
- Symplectic methods still have error. Since it is not in the "energy," it is in the "phase."

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- Double pendulum: later today!

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Thank You

JAMES MADISON UNIVERSITY