

Integrals Approximating π with Non-negative Integrands

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Originally described by Dalzell (1944).

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or

$$\frac{1979}{630} = \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260} = \frac{3959}{1260}.$$

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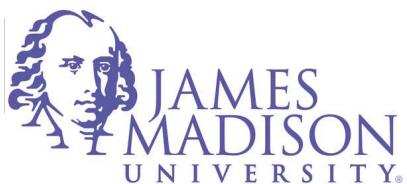
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For nonnegative integers m and n , $\int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$.

So...

Series Expansion (cont'd)

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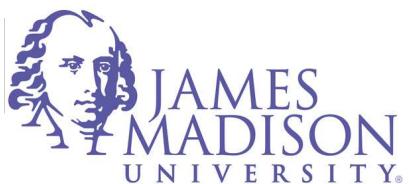
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Roughly three digits of accuracy are added per term.

More General Integrals

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$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = (x^6 - 4x^5 + 5x^4 - 4x^2 + 4) \sum_{k=0}^{n-1} (-4)^{n-1-k} x^{4k} (1-x)^{4k} + \frac{(-4)^n}{1+x^2}$$

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so integrating and simplifying,

$$\frac{(-1)^n}{4^{n-1}} \int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2} dx = \pi - \sum_{k=0}^{n-1} (-1)^k \frac{2^{4-2k} (4k)!(4k+3)!}{(8k+7)!} \times \\ (820k^3 + 1533k^2 + 902k + 165)$$

Pancake Functions

(Backhouse 1995) $I_{m,n} = \int_0^1 \frac{x^m(1-x)^n}{1+x^2} dx = a + b\pi + c \ln(2)$, where rational a, b, c depend on positive integers m and n , and a and b have opposite sign.

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Using $1 < 1+x^2 < 2$ for $x \in (0, 1)$ we get the error bounds

$$\frac{m!n!}{2(m+n+1)!} < a + b = \int_0^1 \frac{x^m(1-x)^n}{1+x^2} dx < \frac{m!n!}{(m+n+1)!}.$$

New Series Expansions

Applying the series expansion process to I_{4n} leads to

$$\pi = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} (-4)^{-nm-k} \int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4) \times x^{4(k+nm)} (1-x)^{4(k+nm)} dx$$

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$n = 1$ leads to previous series expansion.

New Series (cont'd)

With $n = 2$,

$$\pi = \sum_{m=0}^{\infty} 4^{2-2m} \left[\frac{(8m)!(8m+3)!}{(16m+7)!} (6560m^3 + 6132m^2 + 1804m + 165) - \frac{(8m+4)!(8m+7)!}{(16m+15)!} (1640m^3 + 3993m^2 + 3214m + 855) \right]$$

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We can take n as large as we like, and produce series that add roughly $3n$ digits per term!

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Consider the slightly more complicated

$$\int_0^1 \frac{x^m(1-x)^n(a+bx+cx^2)}{1+x^2} dx = \alpha + \beta\pi + \gamma \ln 2,$$

where α, β and γ depends on m, n, a, b and c .

Approximation Algorithm

To set $\int_0^1 \frac{x^m(1-x)^n(a+bx+cx^2)}{1+x^2} dx = z - \pi$ or $\pi - z$ for a given z :

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- Choose m and n as small as possible to make a , b and c as simple as possible.

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$$\int_0^1 \frac{x^{12}(1-x)^{12}(1349-1060x^2)}{38544(1+x^2)} dx = \frac{104348}{33215} - \pi.$$

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Further Directions

- Are other integrands worth considering? I've already tried
$$\int_0^1 \frac{x^m(1-x)^n}{\sqrt{1-x^2}} dx.$$
- Can we approximate other constants this way?