Who Wins When Playing Dreidel

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Outline

- What is Dreidel?
- Past Work
- Markov Chains
- The Pot
- Two Player
- Three Player

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Players drop out if they have to give a counter owning none (or a given number of rounds or Gimels).

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However, it has less glamorous origins, and appears to have originated in sixteenth century England where children played a top spinning game called "teetotal." The game made its way to Germany, and was adopted by Yiddish-speaking Jews.

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BUT this assumed the pot was a continuous variable, and no-one runs out of counters.

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Markov Chains

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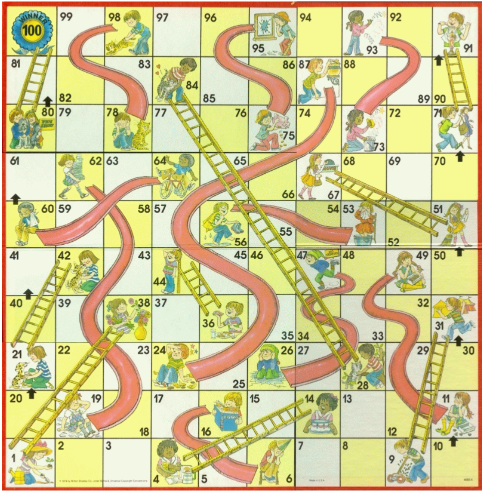
Given a finite number of possible states associated with $1, 2, \ldots, n$, the probability distribution satisfies

$$x^{(t+1)} = x^{(t)} P, \quad p_{ij} = P(X_{t+1} = j | X_t = i).$$

Markov Example – Chutes and Ladders



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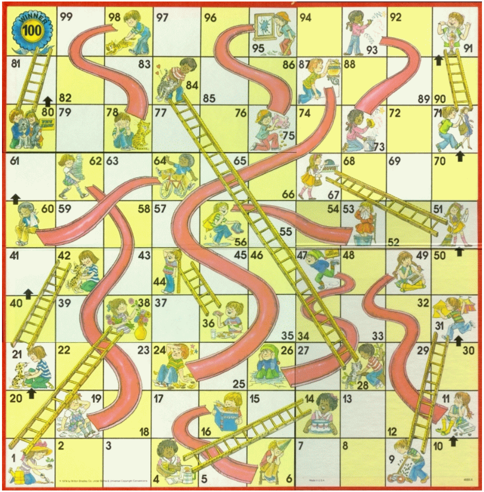

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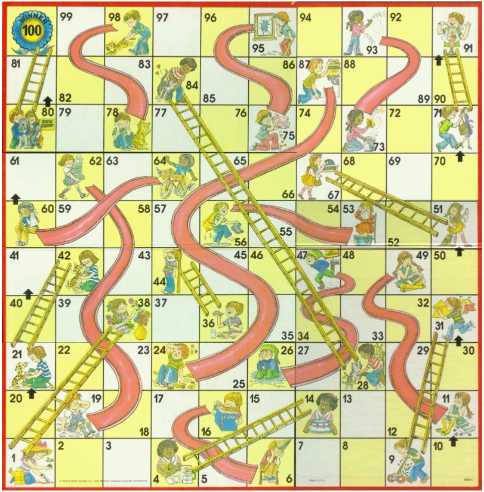

Initially, probability $1/6$ at $(38, 2, 3, 14, 5, 6)$. Probability p at 48, next step probabilities $p/6$ added to $(11, 50, 66, 52, 53, 54)$. $x^{(t+1)} = x^{(t)}P$ with vectors of length 100.

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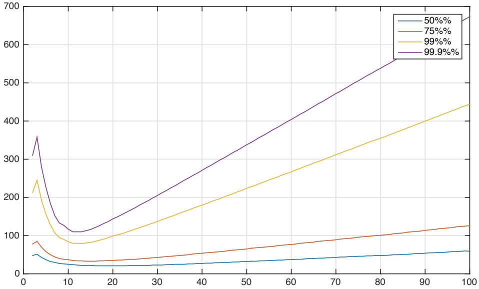
- Best die: Twelve sided.



The Pot

Given N players, $y_i^{(k)}$ probability i counters in the pot before the kth turn, then expected payout at the kth turn is (in order $\mathcal{N}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{S}$)

$$\frac{0}{4} + \frac{1}{4}\sum_i i y_i^{(k)} + \frac{1}{4}\sum_i \left\lceil \frac{i}{2} \right\rceil y_i^{(k)} - \frac{1}{4}$$

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$\mathbf{y}^{(1)}=[0,0,\ldots,0,1]$, the one in the Nth element. Element j contributes $y_j^{(k)}/4$ to $y_j^{(k+1)}$ ($\mathcal{N}$, no payout), $y_N^{(k+1)}$ ($\mathcal{G}$, pot needs to be restarted), $y_{j-\lceil j/2\rceil}^{(k+1)}$ ($\mathcal{H}$, remove half the pot rounded up), and $y_{j+1}^{(k+1)}$ ($\mathcal{S}$, add one to pot). The special case of $\mathcal{H}$ with $j=1$ is equivalent to $\mathcal{G}$.

Expected Payouts per Turn

Turn	Number of players						
	2	3	4	5	6	10	15
1	0.5000	1.0000	1.2500	1.7500	2.0000	3.5000	5.5000
2	0.5625	0.8750	1.1875	1.5000	1.8750	3.1875	4.8125
3	0.5781	0.8906	1.1250	1.4219	1.7344	2.9219	4.4062
4	0.5938	0.8906	1.1055	1.3906	1.6758	2.7617	4.1562
5	0.6025	0.8916	1.1016	1.3809	1.6514	2.6787	4.0244
6	0.6074	0.8928	1.1011	1.3765	1.6384	2.6414	3.9490
7	0.6102	0.8937	1.1009	1.3736	1.6313	2.6259	3.9051
8	0.6118	0.8943	1.1007	1.3718	1.6279	2.6192	3.8818
9	0.6128	0.8947	1.1005	1.3709	1.6264	2.6161	3.8712
10	0.6133	0.8949	1.1004	1.3705	1.6258	2.6144	3.8673
11	0.6136	0.8950	1.1003	1.3703	1.6255	2.6135	3.8665
12	0.6138	0.8950	1.1003	1.3702	1.6254	2.6130	3.8668

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But still assumes large numbers of counters per player. Who wins, and how long does it take?

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Turn number determines who moves, accumulate probability at each turn that the game finishes, and who wins.

Probability Player One Wins

m_2 \ m_1	1	2	3	4	5	6	7
1	0.5441	0.7285	0.8030	0.8459	0.8738	08932	0.9075
2	0.3516	0.5283	0.6340	0.7002	0.7463	0.7801	0.8060
3	0.2555	0.4135	0.5200	0.5939	06481	0.6895	0.7223
4	0.1989	0.3387	0.4401	0.5148	0.5719	0.6170	0.6535
5	0.1629	0.2866	0.3814	0.4541	0.5117	0.5582	0.5967
6	0.1378	0.2484	0.3365	0.4063	0.4629	0.5096	0.5489
7	0.1195	0.2191	0.3010	0.3676	0.4226	0.4688	0.5082

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Other triples (m_1, m_2, p_1): $(9, 10, 0.4789)$, $(10, 10, 0.5057)$, $(14, 15, 0.4863)$, $(15, 15, 0.5038)$, $(19, 20, 0.4899)$, $(20, 20, 0.5028)$.

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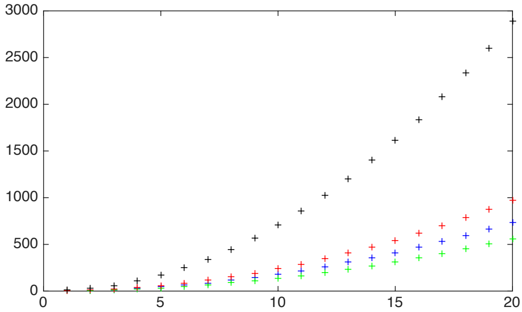

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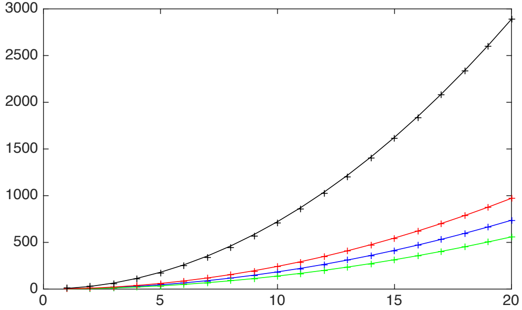

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One fourth of $a(1, i, j, k)$ added to (new) $\mathcal{N}$: $a(2, i, j, k)$, $\mathcal{G}$: $a(2, i+p-1, j-1, k-1)$, $\mathcal{H}$: $a(2, i+\lceil p/2 \rceil, j, k)$, $\mathcal{S}$: $a(2, j-1, j, k)$, with special cases for $\mathcal{H}$ when $p = 1$,

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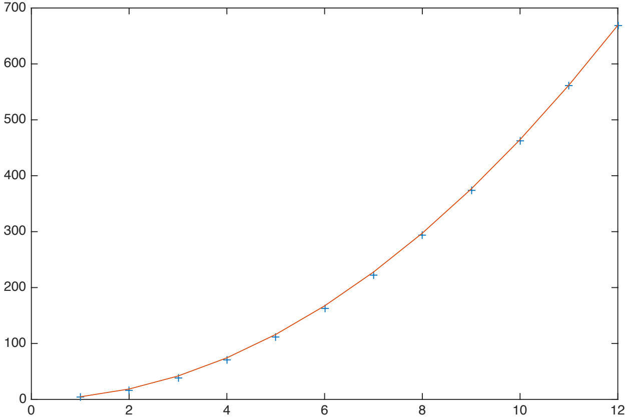
Who Wins With Three (Simulation)

Simulating with 100,000 games:

Counters	Player 1	Player 2	Player 3
1	0.479	0.286	0.235
2	0.398	0.326	0.277
3	0.379	0.327	0.295
4	0.367	0.327	0.306
5	0.362	0.327	0.311
6	0.354	0.332	0.314
7	0.352	0.330	0.318
8	0.350	0.333	0.317
9	0.346	0.333	0.321
10	0.347	0.331	0.322
11	0.346	0.331	0.324
12	0.345	0.332	0.324

Average Length (Three)



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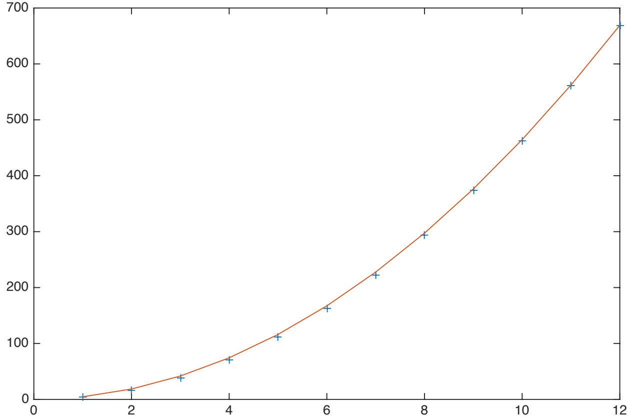


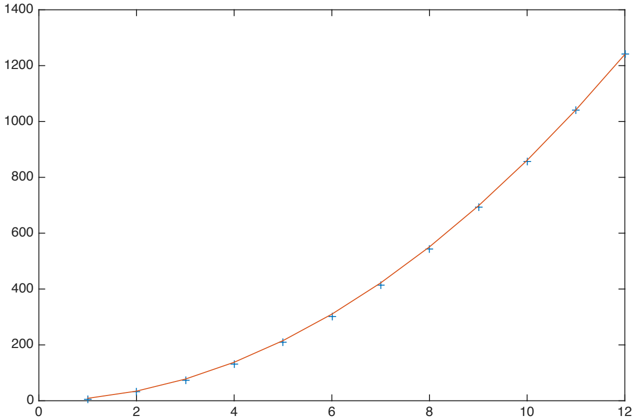
Still appears quadratic, with 5 counters average 112.

Who Wins With Four (Simulation)

Simulating with 400,000 games:

Counters	Player 1	Player 2	Player 3	Player 4
1	0.432	0.271	0.165	0.133
2	0.315	0.277	0.226	0.182
3	0.293	0.263	0.234	0.210
4	0.282	0.260	0.238	0.220
5	0.275	0.259	0.240	0.227
6	0.272	0.256	0.242	0.230
7	0.268	0.256	0.243	0.233
8	0.265	0.255	0.244	0.236
9	0.265	0.254	0.245	0.236
10	0.262	0.253	0.246	0.239
11	0.260	0.253	0.245	0.241
12	0.261	0.254	0.246	0.240

Average Length (Four)



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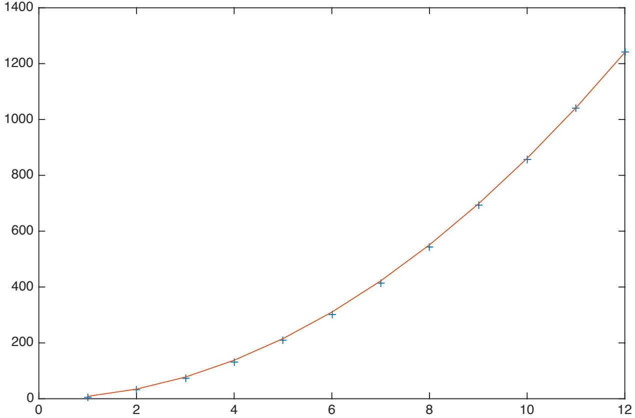
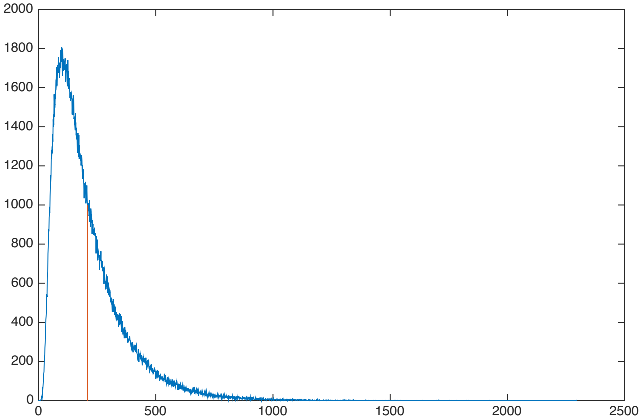


Still appears quadratic, with 5 counters average 208.

Game Length, Four Players, Five Counters Each



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Thank You