Formulas for π

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2653589793238462643383279502884197169399375105820974944592307816406286208998
6280348253421170679821480865132823066470938446095505822317253594081284811174
5028410270193852110555964462294895493038196442881097566593344612847564823378
6783165271201909145648566923460348610454326648213393607260249141273724587006
6063155881748815209209628292540917153643678925903600113305305488204665213841
4695194151160943305727036575959195309218611738193261179310511854807446237996
2749567351885752724891227938183011949129833673362440656643086021394946395224
7371907021798609437027705392171762931767523846748184676694051320005681271452
6356082778577134275778960917363717872146844090122495343014654958537105079227
9689258923542019956112129021960864034418159813629774771309960518707211349999
9983729780499510597317328160963185950244594553469083026425223082533446850352
6193118817101000313783875288658753320838142061717766914730359825349042875546
8731159562863882353787593751957781857780532171226806613001927876611195909216
4201989380952572010654858632788659361533818279682303019520353018529689957736
2259941389124972177528347913151557485724245415069595082953311686172785588907
5098381754637464939319255060400927701671139009848824012858361603563707660104
7101819429555961989467678374494482553797747268471040475346462080466842590694
9129331367702898915210475216205696602405803815019351125338243003558764024749$\cdots$

Outline

A history (and many derivations) of most formulas for π:

- Regular polygon bounds (Archimedes)
- Infinite product with square roots (Viète)
- Infinite product with integers (Wallis)
- Leibniz rule and related infinite series
- Machin's extensions
- Arithmetic-Geometric mean
- A single digit
- The spigot algorithm

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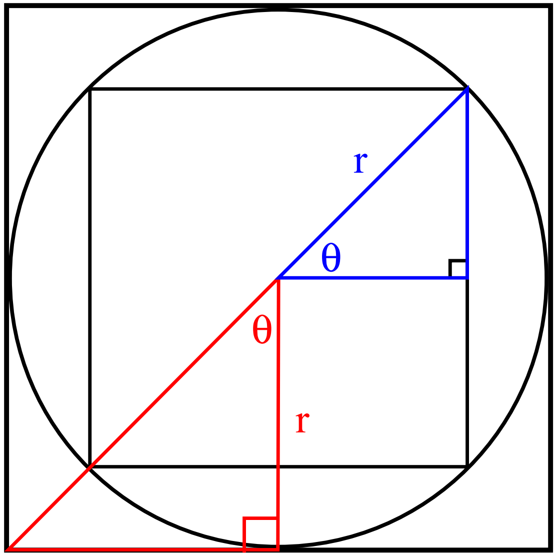

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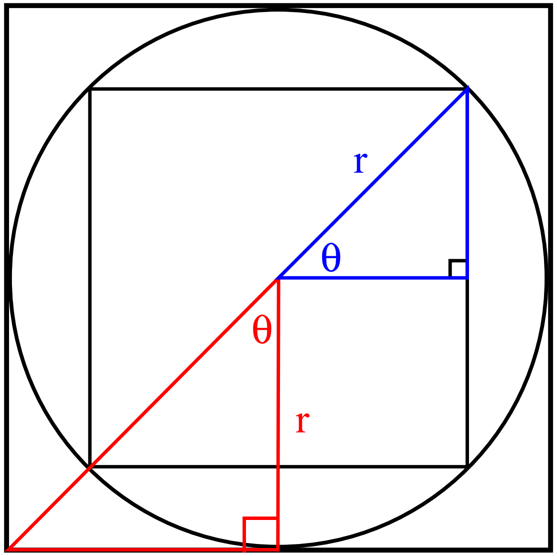

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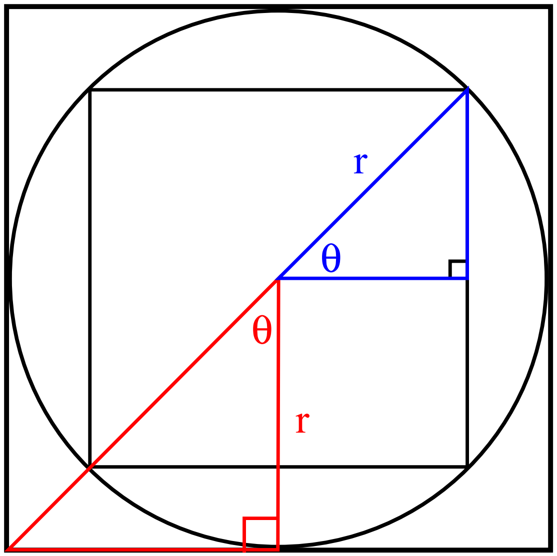
$$n\sin\theta < \pi < n\tan\theta.$$



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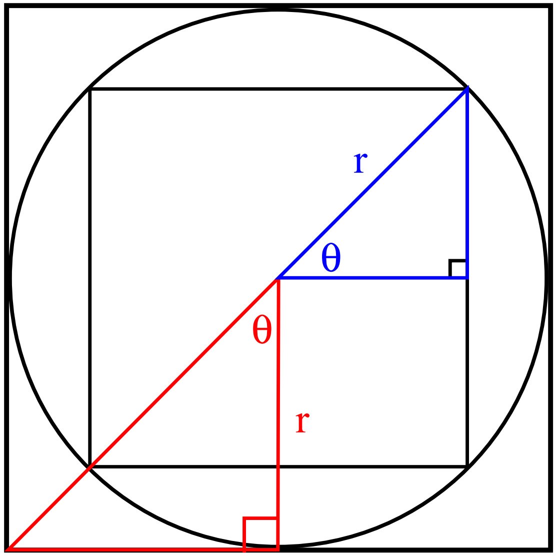

$$n\sin\theta < \pi < n\tan\theta.$$

Doubling number of sides k times, n becomes $2^k n$ and θ becomes $\theta/2^k$:

$$2^k n\sin\left(\frac{\theta}{2^k}\right) < \pi < 2^k n\tan\left(\frac{\theta}{2^k}\right)$$

Doubling Number of Sides

Archimedes chose $n = 6$, so $\theta = \pi/6$, $\sin\theta = 1/2$ and $\tan\theta = 1/\sqrt{3}$. Then $3 < \pi < 6/\sqrt{3} = 2\sqrt{3}$.

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$\sin\dfrac{\theta}{2} = \sqrt{\dfrac{1}{2}\tan\dfrac{\theta}{2}\sin\theta}$.

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Doubling cont'd

Let $\alpha_i = n2^i \sin(\theta/2^i)$, $\beta_i = n2^i \tan(\theta/2^i)$, then $\alpha_0 = 3$, $\beta_i = 2\sqrt{3}$, and

$$\beta_{i+1} = \frac{2\alpha_i\beta_i}{\alpha_i + \beta_i} \quad \text{and} \quad \alpha_{i+1} = \sqrt{\alpha_i\beta_{i+1}}.$$

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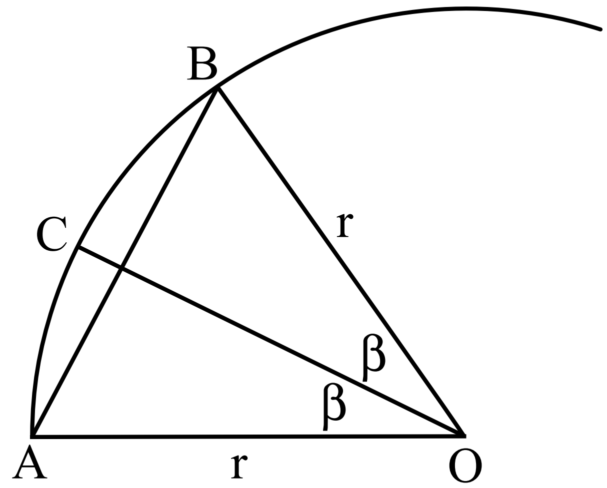
Viète's Polygon (1593)

The area of an n-sided regular polygon is

$$A(n) \quad = \quad n \times \text{area} \triangle OAB$$



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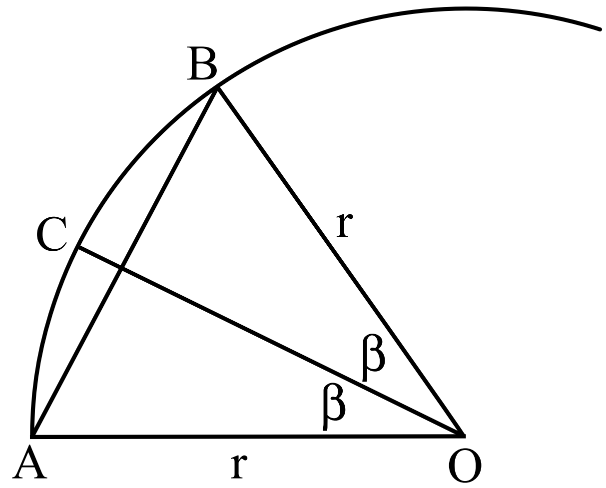

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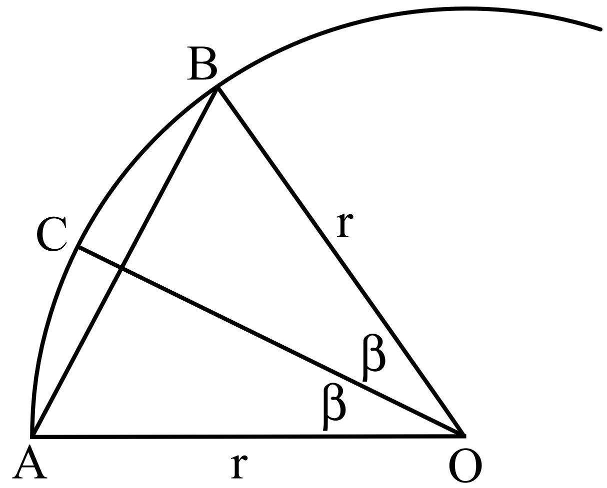

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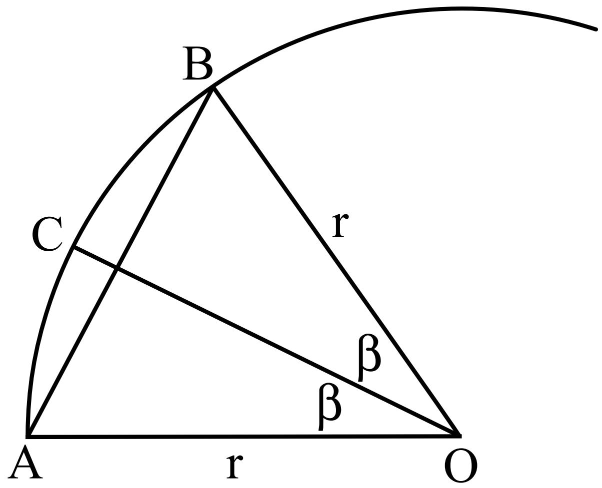

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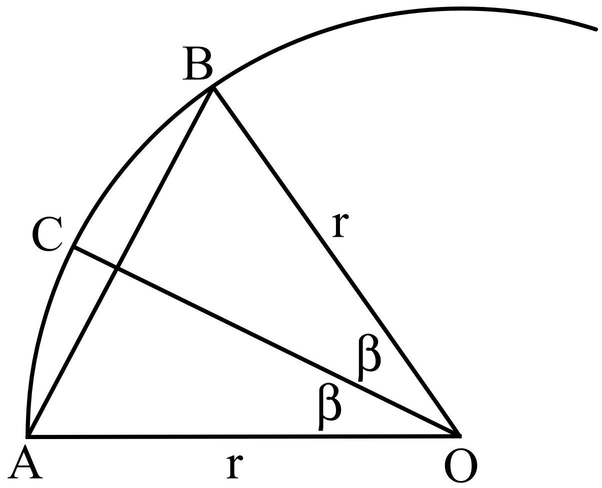

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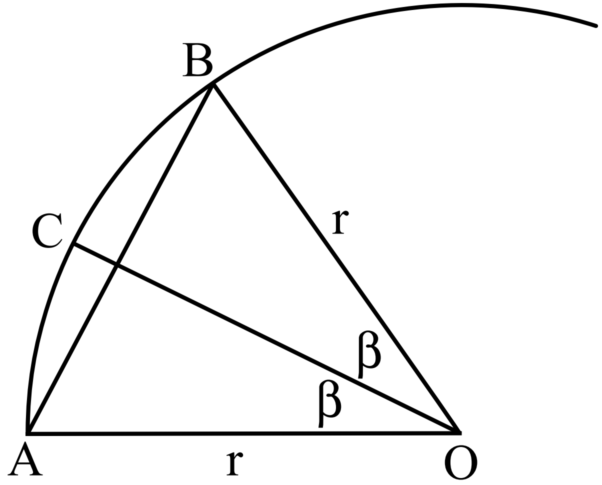

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$$\frac{\left(\frac{1}{nr^2\sin 2\beta}\right)}{\pi r^2} = \cos\beta\cos\frac{\beta}{2}\cos\frac{\beta}{2^2}\cdots \quad \text{or} \quad \pi = \frac{n\sin 2\beta}{2\cos\beta\cos\frac{\beta}{2}\cos\frac{\beta}{2^2}\cdots}.$$

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The first infinite product in mathematics, but converges slowly and all those square roots are difficult.

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&= \cdots \\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}.
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&= \cdots \\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}\underbrace{\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}_{\to 1}.
\end{aligned}$$

Euler's Generalization

By applying $\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$ repeatedly,

$$\begin{aligned}
\frac{\sin\theta}{\theta} &= \cos\frac{\theta}{2}\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{4}\frac{\sin\frac{\theta}{4}}{\frac{\theta}{4}} \\
&= \cdots \\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}\underbrace{\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}_{\to 1}.
\end{aligned}$$

Set $\theta = \pi/2$ to get Viète's result.

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Using recurrence relations (integration by parts twice),

$$I_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}.$$

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$$\frac{I_{2n}}{I_{2n+1}} = \frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{3\cdot 5\cdot 7\cdots(2n+1)}{2\cdot 4\cdot 6\cdots(2n)}$$

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Wallis (cont'd)

Now on $[0, \pi/2]$, $0 \leq \sin x \leq 1$ so $\sin^{m+1} x \leq \sin^m x$ and $I_{m+1} \leq I_m$.

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So as $n \to \infty$, $\dfrac{I_{2n}}{I_{2n+1}} \to 1$ and $\dfrac{\pi}{2} = \displaystyle\prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}$.

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Still converges slowly, but at least no square roots.

Calculus Beginnings

The geometric series $a + ar + ar^2 + \cdots + ar^n$ has sum $\dfrac{a(1 - r^{n+1})}{(1 - r)}$.

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arctan Series

Integrate both sides with respect to t from 0 to x where $-1 \leq x \leq 1$:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt.$$

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Leibniz & Gregory (independently 1670) set $x = 1$:

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Converges very slowly (5000 terms for 3 digits).

Better Series

Sharp (1699) set $x = \pi/3$, and found 71 digits:

$$\frac{\pi}{6} \quad = \quad \frac{1}{\sqrt{3}} - \frac{(1/\sqrt{3})^3}{3} + \frac{(1/\sqrt{3})^5}{5} - \frac{(1/\sqrt{3})^7}{7} + \cdots$$

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Newton (1676, published much later) expanded $1/\sqrt{1-x^2}$ and integrated:

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots + \frac{(2n)!}{2^{2n}(n!)^2}\frac{x^{2n+1}}{2n+1},$$

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and with $x = 1/2$,

$$\frac{\pi}{3} = 1 + \frac{1}{24} + \frac{3}{640} + \cdots + \frac{(2n)!}{2^{4n}(n!)^2(2n+1)} + \cdots.$$

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Euler's arctan (cont'd)

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$$\int_0^1 \frac{s^n}{2\sqrt{1-s}}\, ds \quad = \quad \int_0^{\pi/2} \frac{\sin^{2n}\theta}{2\sqrt{1-\sin^2\theta}} 2\sin\theta\cos\theta\, d\theta$$

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Euler's arctan (cont'd)

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$$\begin{aligned}\int_0^1 \frac{s^n}{2\sqrt{1-s}}\,ds &= \int_0^{\pi/2} \frac{\sin^{2n}\theta}{2\sqrt{1-\sin^2\theta}} 2\sin\theta\cos\theta\,d\theta \\ &= \int_0^{\pi/2} \sin^{2n+1}\theta\,d\theta = \frac{2^{2n}(n!)^2}{(2n+1)!}.\end{aligned}$$

So $\arctan(x) = \dfrac{x}{1+x^2}\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{x^2}{1+x^2}\right)^n$,

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$$\pi = 2\sum_{n=0}^{\infty} \frac{2^n(n!)^2}{(2n+1)!} = 2\left(1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots\right).$$

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This converges very fast.

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Double angle formulas give $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ and $\tan 4\theta = \dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$.

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Since $\tan(x+y) = \dfrac{\tan x + \tan y}{1-\tan x\tan y}$ and $\tan(\pi/4) = 1$,

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So

$$\arctan\left(\frac{1}{239}\right) = 4\theta - \frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \frac{\pi}{4}.$$

Machin (cont'd)

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

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Machin (cont'd)

$$
\begin{aligned}
\frac{\pi}{4} &= 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \\
&= 4\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \cdots\right) - \\
&\qquad \left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7} + \cdots\right)
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&\qquad \left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7} + \cdots\right) \\
&= \frac{4}{5}\left(1 - \frac{1}{3}\frac{4}{100} + \frac{1}{5}\frac{4^2}{100^2} - \frac{1}{7}\frac{4^3}{100^3} + \cdots\right) - \\
&\qquad \frac{1}{239}\left(1 - \frac{1}{3}\frac{1}{57121} + \frac{1}{5}\frac{1}{57121^2} - \frac{1}{7}\frac{1}{57121^3} + \cdots\right).
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First series is relatively easy in decimals, and the second converges very quickly.

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First series is relatively easy in decimals, and the second converges very quickly.

Machin (1706) found 100 decimal places.

Euler (again!)

Since $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$,

$$\tan\left(\arctan\left(\frac{1}{p}\right) - \arctan\left(\frac{1}{p+q}\right)\right) \quad = \quad \frac{\frac{1}{p} - \frac{1}{p+q}}{1 + \frac{1}{p}\frac{1}{p+q}}$$

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so

$$\arctan\left(\frac{1}{p}\right) = \arctan\left(\frac{1}{p+q}\right) + \arctan\left(\frac{q}{p^2+pq+1}\right).$$

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Euler (cont'd)

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- With $p = 2$, $q = 1$, $\arctan(1/2) = \arctan(1/3) + \arctan(1/7)$, so $\pi/4 = 2\arctan(1/3) + \arctan(1/7)$.

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- With $p = 3$, $q = 4$, $\arctan(1/3) = \arctan(1/7) + \arctan(2/11)$, so $\pi/4 = 3\arctan(1/7) + 2\arctan(2/11)$.

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- With $p = 3$, $q = 4$, $\arctan(1/3) = \arctan(1/7) + \arctan(2/11)$, so $\pi/4 = 3\arctan(1/7) + 2\arctan(2/11)$.
- With $p = 11/2$, $q = 3/2$, $\arctan(2/11) = \arctan(1/7) + \arctan(3/79)$, so $\pi/4 = 5\arctan(1/7) + 2\arctan(3/79)$.

Euler (cont'd)

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Using his formula $\left(x = \frac{1}{7} \to \frac{x^2}{1+x^2} = \frac{2}{100},\ x = \frac{3}{79} \to \frac{x^2}{1+x^2} = \frac{144}{10^5}\right)$,

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$$\begin{aligned}\frac{\pi}{4} &= \frac{7}{10}\left(1 + \frac{2}{3}\frac{2}{100} + \frac{2\cdot 4}{3\cdot 5}\frac{2^2}{100^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\frac{2^3}{100^3} + \cdots\right) \\ &\quad + \frac{7584}{10^5}\left(1 + \frac{2}{3}\frac{144}{10^5} + \frac{2\cdot 4}{3\cdot 5}\frac{144^2}{10^{10}} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\frac{144^3}{10^{15}} + \cdots\right).\end{aligned}$$

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Euler (cont'd)

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Great for decimal calculation, Euler found 20 digits in one hour by hand!

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Machin-like Formulae

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- Ferguson (1945) used $\pi/4 = 3\arctan(1/4) + \arctan(1/20) + \arctan(1/1985)$ to find 528 digits – last hand calculation.
- Guillard & Boyer (1974) used $\pi/4 = 12\arctan(1/18) + 8\arctan(1/57) - 5\arctan(1/239)$ and $\pi/4 = 6\arctan(1/8) + 2\arctan(1/57) + \arctan(1/239)$ to first break a million digits.

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Yet more series

There are hundreds of series approximations to π. (Google "pi formulas" check out the Mathworld website). Using modular forms:

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- Borwein & Borwein (1993): $\dfrac{1}{\pi} = 12 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{(n!)^3 (3n)!(640320^3)^{n+1/2}}$,

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Lucas (2008): using series expansions related to $\displaystyle\int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}\,dx$, formed series where each term can give an arbitrary number of digits.

Arithmetic-Geometric Mean

Given $a_0 = a \geq b_0 = b > 0$, let $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = \sqrt{a_n b_n}$.

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so convergence is quadratic, doubling the number of similar digits at each step. Call this limit $M(a, b)$.

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$$\sum_{n=0}^{\infty} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \left(\frac{1}{16} \right)^n$$

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Proof relies upon $(k<8)$

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Hundreds more have been found of similar form.

BBP for Digit Extraction

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To find the fractional part (and particularly the $(d+1)$st digit) use modular arithmetic to quickly evaluate the first sum, and use a few terms of the second to ensure the first digit is accurate:

$$\text{frac}(16^d S_1) = \sum_{n=0}^{d} \frac{16^{d-n} \text{ mod } 8n+1}{8n+1} \text{ mod } 1 + \sum_{n=d+1}^{\infty} \frac{16^{d-n}}{8n+1} \text{ mod } 1.$$

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Euler's formula is thus $2.22222\ldots$ in a variable base representation.

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Spigot algorithm? Exact arithmetic produces successive digits of π.

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Conclusion

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The primary purpose of the DATA statement is to give names to constants; instead of referring to pi as 3.141592653589793 at every appearance, the variable PI can be given that value with a DATA statement and used instead of the longer form of the constant. This also simplifies modifying the program, should the value of pi change.

– FORTRAN manual for Xerox Computers

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