Simple Heteroclinic Orbit Examples in the Plane

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July 4, 2012

9th AIMS Conference on Dynamical Systems, Differential Equations and Applications
Special Session 10

Outline

- Planar Systems
- Heteroclinic Orbits
- First ode, spirals, every point on a heteroclinic orbit
- Second ode, heteroclinic limit points along a line
- Power Series Method for odes

Planar Systems

$\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ for real polynomials $P(x, y)$, $Q(x, y)$ in x and y form a polynomial differential system in the plane, and have been extensively studied over the decades.

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Over a thousand papers on quadratic systems alone, with a bibliography compiled by the Delft University of Technology (1904-1997)

Heteroclinic Orbits

A heteroclinic orbit is a solution to the system of odes $\dot{x} = f(t, x)$ where $x \longrightarrow x_a$ as $t \longrightarrow \infty$ and $x \longrightarrow x_b$ as $t \longrightarrow -\infty$ for given points x_a, x_b.

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Numerically locating heteroclinic orbits (if they exist) is challenging, and often reduces to solving an infinite boundary value problem.

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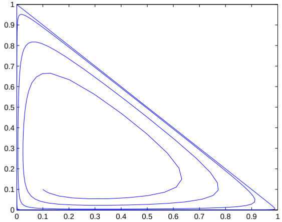

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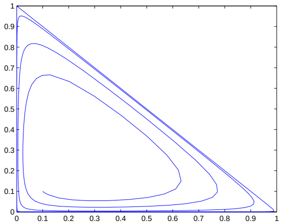

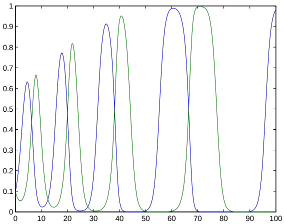

First ODE

Consider the system

$$\begin{aligned}\dot{x} &= a(x^2 - y^2) - 2bxy + cx - dy + e, \\ \dot{y} &= b(x^2 - y^2) + 2axy + dx + cy + f,\end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= g, \\ y(0) &= h,\end{aligned}$$

for given constants a to h.

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With $z(t) = x(t) + iy(t)$,

$$\dot{z} = (a + ib)z^2 + (c + id)z + (e + if) \quad \text{with} \quad z(0) = g + ih,$$

with all constants a to h being real.

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By completing the square and scaling by $a + ib$, any quadratic complex ode can be reduced to

$$\dot{z} = z^2 + (a + ib) \quad \text{with} \quad z(0) = c + id,$$

where a, b, c and d are real constants.

Analytic Solution

Let $e = \dfrac{\sqrt{a+\sqrt{a^2+b^2}}}{\sqrt{2}}$ and $f = \dfrac{b}{\sqrt{2\left(a+\sqrt{a^2+b^2}\right)}}$ so

$\sqrt{a+ib} = \pm(e+if)$ and $\sqrt{-(a+ib)} = \pm(-f+ie)$.

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$$z = (e + if)\tan((e + if)t + C), \; C = \arctan\left(\frac{c + id}{e + if}\right) = g + ih.$$

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Using the definitions of complex arctan, log, sin and cos, we get

$$z(t) = \frac{e\sin(2(et+g)) - f\sinh(2(ft+h))}{\cosh(2(ft+h))+\cos(2(et+g))} + i\frac{f\sin(2(et+g)) + e\sinh(2(ft+h))}{\cosh(2(ft+h))+\cos(2(et+g))}.$$

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Then $z(t) = x(t) + iy(t) = \dfrac{-\left(t + \frac{c}{c^2+d^2}\right) - i\frac{d}{c^2+d^2}}{\left(t + \frac{c}{c^2+d^2}\right)^2 + \left(\frac{d}{c^2+d^2}\right)^2}$.

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If we let $m = c/(c^2 + d^2)$ and $n = d/(c^2 + d^2)$ then $x(t) = -\dfrac{t + m}{(t + m)^2 + n^2}$ and $y(t) = -\dfrac{n}{(t + m)^2 + n^2}$.

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$(x(t),y(t))\to(0,0)$ as $t\to\pm\infty$, so the solution starting from any point is on a homoclinic orbit, with the same homoclinic point.

Zero Constant Orbits

$$x^2 + y^2 = \frac{(t+m)^2 + n^2}{((t+m)^2 + n^2)^2} = \frac{1}{(t+m)^2 + n^2} = -\frac{y}{n},$$

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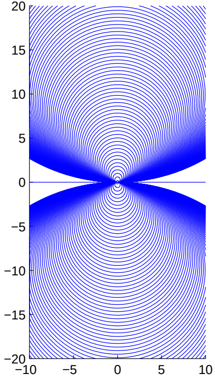

Positive Real Constant

If $b = 0$ and $a > 0$, then $e = \sqrt{a}$, $f = 0$ and

$$z(t) = x(t) + iy(t) = \sqrt{a} \cdot \frac{\sin(2(\sqrt{a}t + g)) + i \sinh(2h)}{\cosh(2h) + \cos(2(\sqrt{a}t + g))}.$$

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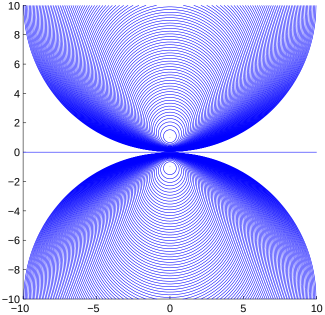

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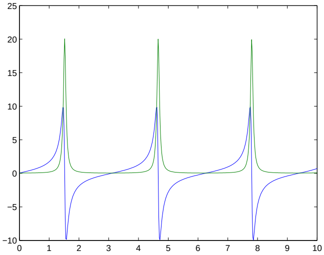
Velocity small near the origin, can become very large away from the origin. $\dot{z} = z^2 + 1$, $z(0) = 0.05 + 0.05i$:



Negative Real Constant

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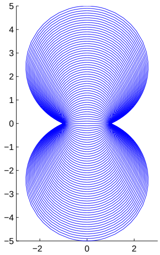

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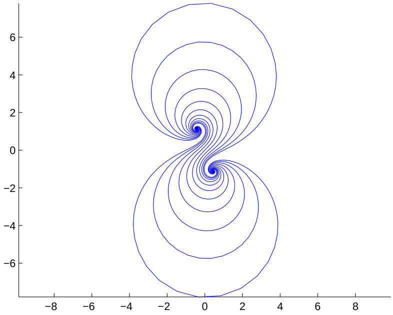
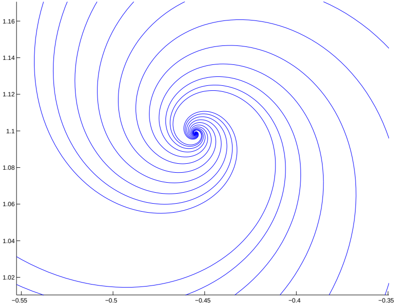
Orbits cannot be represented as algebraic equations in x and y only, and are spirals similar to Carnu or Euler spirals, with exponential convergence for large magnitude t.

First Example

$$\dot{z} = z^2 + (1 + i),\ z(0) = -1, -0.8, \ldots, 1.$$





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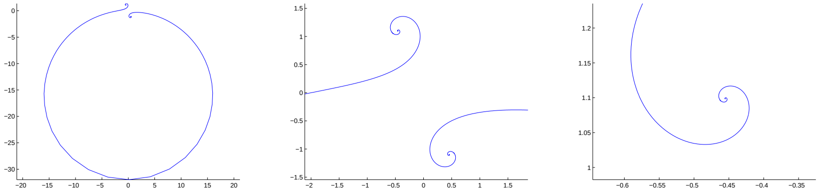

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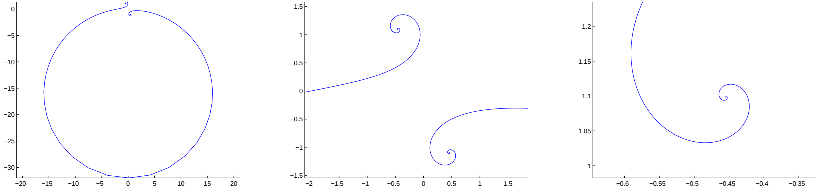
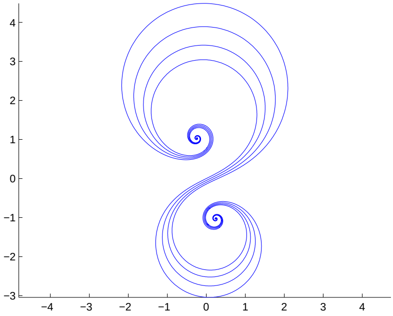
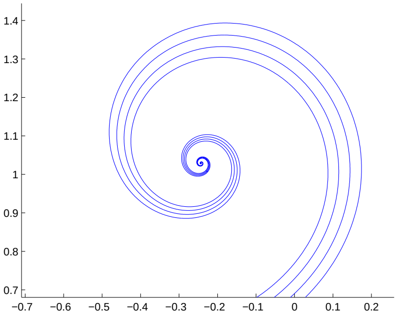


Same exponential convergence after more dramatic intermediate circular curve.

Second Example

$\dot{z} = z^2 + 1 + i/2$, with $z(0) = 0, 0.1, 0.2, 0.3$.





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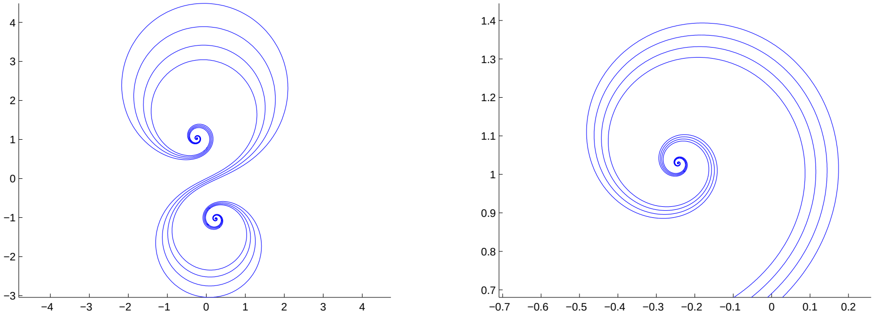

The smaller the ratio a/b, the faster the convergence of the spiral.

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The line $-ax - by = 0$ could be called a heteroclinic line.

Analytic Solution

Starting with $\dot{x} = -y(1 - ax - by)$ and $\dot{y} = x(1 - ax - by)$ with $x(0) = c$, $y(0) = d$, let $r = \sqrt{c^2 + d^2}$, $x(t) = r\cos(\theta(t))$ and $y(t) = r\sin(\theta(t))$.

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$$\theta(t) = -2\arctan\left(\frac{-br + \tanh\left(\frac{t+C}{2}\sqrt{r^2(a^2+b^2)-1}\right)\sqrt{r^2(a^2+b^2)-1}}{1+ar}\right)$$

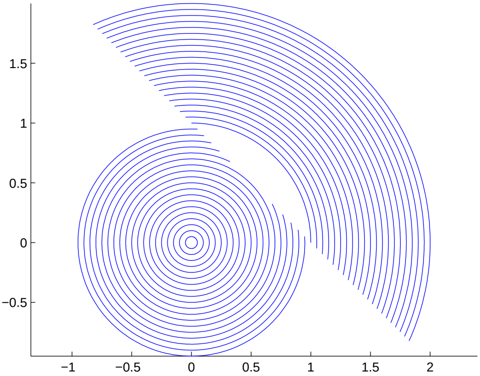
when $r^2 > 1/(a^2 + b^2)$, and

$$\theta(t) = -2\arctan\left(\frac{-br - \tan\left(\frac{t+C}{2}\sqrt{1-r^2(a^2+b^2)}\right)\sqrt{1-r^2(a^2+b^2)}}{1+ar}\right) + 2k\pi$$

when $r^2 < 1/(a^2 + b^2)$, and k is an integer chosen to ensure $\theta(t)$ stays continuous and monotonic.

Example

$\dot{x} = -y(1-x-y)$ and $\dot{y} = x(1-x-y)$ with $d = 0$ and $c = 0.05, 0.1, 0.15, \ldots, 2$:



More General Case

$\dot{x} = -yf(x, y)$, $\dot{y} = xf(x, y)$ for any function $f(x, y)$ has arcs of circles as orbits, with the solutions of $f(x, y) = 0$ as heteroclinic lines.

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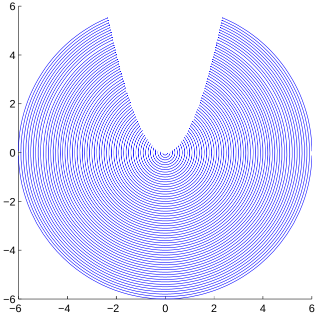
For example, $x' = -y(y - x^2)$ and $y' = x(y - x^2)$ with $x(0) = 0$ and $y(0) = -0.1, -0,2, -0.3, \ldots, -6$.



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- No transcendental function evaluation, so is much faster.

JAMES MADISON UNIVERSITY

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