Taylor Series Without High Order Differentiation

Stephen Lucas

Department of Mathematics and Statistics

James Madison University, Harrisonburg VA

Outline

- Define Taylor series and issues.

Outline

- Define Taylor series and issues.
- Replace function by “simple” algebraic or differential equations and use formal series substitution.

JAMES MADISON UNIVERSITY

Outline

- Define Taylor series and issues.
- Replace function by “simple” algebraic or differential equations and use formal series substitution.
- Straightforward Examples.

Outline

- Define Taylor series and issues.
- Replace function by “simple” algebraic or differential equations and use formal series substitution.
- Straightforward Examples.
- A new Bernoulli number algorithm.

JAMES MADISON UNIVERSITY

Outline

- Define Taylor series and issues.
- Replace function by "simple" algebraic or differential equations and use formal series substitution.
- Straightforward Examples.
- A new Bernoulli number algorithm.
- Reciprocals and general powers of functions.

JAMES MADISON UNIVERSITY

Outline

- Define Taylor series and issues.
- Replace function by “simple” algebraic or differential equations and use formal series substitution.
- Straightforward Examples.
- A new Bernoulli number algorithm.
- Reciprocals and general powers of functions.
- The inverse of a polynomial.

JAMES MADISON UNIVERSITY

Taylor Series

Assuming f is sufficiently differentiable,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

Taylor Series

Assuming f is sufficiently differentiable,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

In simple cases, derivative are easy to find. In others, it can be horrendous.

Taylor Series

Assuming f is sufficiently differentiable,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

In simple cases, derivative are easy to find. In others, it can be horrendous. For example, if $f(x) = \sin\sqrt{x^2-1}$, then

$$f^{(4)}(x) = \sin\sqrt{x^2-1}\left(\frac{x^4}{(x^2-1)^2} - \frac{15x^4}{(x^2-1)^3} + \frac{18x^2}{(x^2-1)^2} - \frac{3}{x^2-1}\right) + \cos\sqrt{x^2-1}\left(\frac{6x^4}{(x^2-1)^{5/2}} - \frac{6x^2}{(x^2-1)^{3/2}} - \frac{15x^4}{(x^2-1)^{7/2}} + \frac{18x^2}{(x^2-1)^{5/2}} - \frac{3}{(x^2-1)^{3/2}}\right),$$

Taylor Series

Assuming f is sufficiently differentiable,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

In simple cases, derivative are easy to find. In others, it can be horrendous. For example, if $f(x) = \sin\sqrt{x^2-1}$, then

$$f^{(4)}(x) = \sin\sqrt{x^2-1}\left(\frac{x^4}{(x^2-1)^2} - \frac{15x^4}{(x^2-1)^3} + \frac{18x^2}{(x^2-1)^2} - \frac{3}{x^2-1}\right) + \cos\sqrt{x^2-1}\left(\frac{6x^4}{(x^2-1)^{5/2}} - \frac{6x^2}{(x^2-1)^{3/2}} - \frac{15x^4}{(x^2-1)^{7/2}} + \frac{18x^2}{(x^2-1)^{5/2}} - \frac{3}{(x^2-1)^{3/2}}\right),$$

and the tenth derivative has 45 terms.

JAMES MADISON UNIVERSITY

Taylor Series

Assuming f is sufficiently differentiable,

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots . \end{aligned}$$

In simple cases, derivative are easy to find. In others, it can be horrendous. For example, if $f(x) = \sin\sqrt{x^2-1}$, then

$$f^{(4)}(x) = \sin\sqrt{x^2-1}\left(\frac{x^4}{(x^2-1)^2} - \frac{15x^4}{(x^2-1)^3} + \frac{18x^2}{(x^2-1)^2} - \frac{3}{x^2-1}\right)$$
$$+ \cos\sqrt{x^2-1}\left(\frac{6x^4}{(x^2-1)^{5/2}} - \frac{6x^2}{(x^2-1)^{3/2}} - \frac{15x^4}{(x^2-1)^{7/2}} + \frac{18x^2}{(x^2-1)^{5/2}} - \frac{3}{(x^2-1)^{3/2}}\right),$$

and the tenth derivative has 45 terms. Also Maple “hangs” when trying to solve $y'' = \sin(y)$ using series after about 8 terms.

JAMES MADISON UNIVERSITY

An Ode Solution Technique

A first course in differential equations introduces a power series substitution method for second order linear differential equations of the from $p(x)y'' + q(x)y' + r(x)y = f(x)$, as long as p, q, r, f are sufficiently simple.

An Ode Solution Technique

A first course in differential equations introduces a power series substitution method for second order linear differential equations of the from $p(x)y'' + q(x)y' + r(x)y = f(x)$, as long as p, q, r, f are sufficiently simple.

The method of Frobenius multiplies each power series by x^r to deal with expansions around regular singular points (e.g. Bessel's odes).

JAMES MADISON UNIVERSITY

An Ode Solution Technique

A first course in differential equations introduces a power series substitution method for second order linear differential equations of the from $p(x)y'' + q(x)y' + r(x)y = f(x)$, as long as p, q, r, f are sufficiently simple.

The method of Frobenius multiplies each power series by x^r to deal with expansions around regular singular points (e.g. Bessel's odes).

Almost never seen again, especially when an ode is nonlinear. But, there is no reason why it can't be applied to these odes with some alterations, particularly those whose solution is a function whose Taylor series we want...

JAMES MADISON UNIVERSITY

Solution Approach

The power series substitution method works because the left and right hand sides can be expanded as power series, and coefficients of successive powers of x equated.

Solution Approach

The power series substitution method works because the left and right hand sides can be expanded as power series, and coefficients of successive powers of x equated.

Returning to finding the Taylor series of f (assuming it is analytic at $x = 0$), rewrite $y = f(x)$ as the first order equation $y' = f'(x)$ with $y(0) = f(0)$.

JAMES MADISON UNIVERSITY

Solution Approach

The power series substitution method works because the left and right hand sides can be expanded as power series, and coefficients of successive powers of x equated.

Returning to finding the Taylor series of f (assuming it is analytic at $x = 0$), rewrite $y = f(x)$ as the first order equation $y' = f'(x)$ with $y(0) = f(0)$.

Add additional variables and their derivatives in such a way that we get a system of first order equations of the form $Y' = F(Y)$, where each right hand side is polynomial in the variables and x.

JAMES MADISON UNIVERSITY

Solution Approach

The power series substitution method works because the left and right hand sides can be expanded as power series, and coefficients of successive powers of x equated.

Returning to finding the Taylor series of f (assuming it is analytic at $x = 0$), rewrite $y = f(x)$ as the first order equation $y' = f'(x)$ with $y(0) = f(0)$.

Add additional variables and their derivatives in such a way that we get a system of first order equations of the form $Y' = F(Y)$, where each right hand side is polynomial in the variables and x. In some cases, we may multiply both sides of an equation by a power of a variable so that both left and right hand sides are simple power series.

JAMES MADISON UNIVERSITY

Solution Approach

The power series substitution method works because the left and right hand sides can be expanded as power series, and coefficients of successive powers of x equated.

Returning to finding the Taylor series of f (assuming it is analytic at $x = 0$), rewrite $y = f(x)$ as the first order equation $y' = f'(x)$ with $y(0) = f(0)$.

Add additional variables and their derivatives in such a way that we get a system of first order equations of the form $Y' = F(Y)$, where each right hand side is polynomial in the variables and x. In some cases, we may multiply both sides of an equation by a power of a variable so that both left and right hand sides are simple power series.

We call the class of functions that can be formulated in this way Picardable.

JAMES MADISON UNIVERSITY

Why Picardable?

Ed Parker and Jim Sochacki showed that the system $Y' = F(Y)$ where the right hands sides are polynomial in the variables could be solved as Taylor series using Picard's method around a point where the functions are analytic.

JAMES MADISON UNIVERSITY

Why Picardable?

Ed Parker and Jim Sochacki showed that the system $Y' = F(Y)$ where the right hands sides are polynomial in the variables could be solved as Taylor series using Picard's method around a point where the functions are analytic.

The functions that could be written this way are called projectively polynomial (pp), and were shown to be a strict subset of the class of analytic functions.

Why Picardable?

Ed Parker and Jim Sochacki showed that the system $Y' = F(Y)$ where the right hands sides are polynomial in the variables could be solved as Taylor series using Picard's method around a point where the functions are analytic.

The functions that could be written this way are called projectively polynomial (pp), and were shown to be a strict subset of the class of analytic functions.

Later, Paul Warne showed that the same Taylor series could be obtained by formal power substitution,

JAMES MADISON UNIVERSITY

Why Picardable?

Ed Parker and Jim Sochacki showed that the system $Y' = F(Y)$ where the right hands sides are polynomial in the variables could be solved as Taylor series using Picard's method around a point where the functions are analytic.

The functions that could be written this way are called projectively polynomial (pp), and were shown to be a strict subset of the class of analytic functions.

Later, Paul Warne showed that the same Taylor series could be obtained by formal power substitution, and David Carothers showed that every pp system could be rewritten so that only quadratic polynomial terms are required, so only Cauchy products of power series are needed.

JAMES MADISON UNIVERSITY

Picardable Versus Pp

Consider $y = (e^x - 1)/x$, which is analytic at zero.

Picardable Versus Pp

Consider $y = (e^x - 1)/x$, which is analytic at zero.

After years of trying, it hasn't been shown to be pp, but still no proof either way.

Picardable Versus Pp

Consider $y = (e^x - 1)/x$, which is analytic at zero.

After years of trying, it hasn't been shown to be pp, but still no proof either way.

Recently I think I've shown $J_0(x)$ is not pp. (Ed still needs convincing $\ddot\smile$)

JAMES MADISON UNIVERSITY

Picardable Versus Pp

Consider $y = (e^x - 1)/x$, which is analytic at zero.

After years of trying, it hasn't been shown to be pp, but still no proof either way.

Recently I think I've shown $J_0(x)$ is not pp. (Ed still needs convincing $\ddot\smile$)

But consider $xy' = 1 - y + z$ and $z' = z + 1$ with $y(0) = 1$ and $z(0) = 0$.

JAMES MADISON UNIVERSITY

Picardable Versus Pp

Consider $y = (e^x - 1)/x$, which is analytic at zero.

After years of trying, it hasn't been shown to be pp, but still no proof either way.

Recently I think I've shown $J_0(x)$ is not pp. (Ed still needs convincing $\ddot\smile$)

But consider $xy' = 1 - y + z$ and $z' = z + 1$ with $y(0) = 1$ and $z(0) = 0$. Then $z = e^x - 1$ and $y = (e^x - 1)/x$. So y is Picardable (and can be solved by Picard's method or power series substitution), while it might not be pp.

JAMES MADISON UNIVERSITY

Picardable Versus Pp

Consider $y = (e^x - 1)/x$, which is analytic at zero.

After years of trying, it hasn't been shown to be pp, but still no proof either way.

Recently I think I've shown $J_0(x)$ is not pp. (Ed still needs convincing $\ddot\smile$)

But consider $xy' = 1 - y + z$ and $z' = z + 1$ with $y(0) = 1$ and $z(0) = 0$. Then $z = e^x - 1$ and $y = (e^x - 1)/x$. So y is Picardable (and can be solved by Picard's method or power series substitution), while it might not be pp.

Conjecture: Picardable equals those analytic functions whose power series hold a finite amount of information.

Some Simple Examples

- If $y = e^x$

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$.

JAMES MADISON UNIVERSITY

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

JAMES MADISON UNIVERSITY

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$.

JAMES MADISON UNIVERSITY

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. Equate powers of x to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots$. Equate powers of x to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.

- $y = \sin x$ satisfies $y'' = -y$,

JAMES MADISON UNIVERSITY

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$.
Equate powers of x to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.

- $y = \sin x$ satisfies $y'' = -y$, or better to add $z = \cos x$, then $y' = z$ and $z' = -y$ with $y(0) = 0$, $z(0) = 1$.

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. Equate powers of x to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.

- $y = \sin x$ satisfies $y'' = -y$, or better to add $z = \cos x$, then $y' = z$ and $z' = -y$ with $y(0) = 0$, $z(0) = 1$. Let $y = \sum_{k=1}^{\infty} a_k x^k$ and $z = 1 + \sum_{k=1}^{\infty} b_k x^k$.

JAMES MADISON UNIVERSITY

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. Equate powers of x to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.

- $y = \sin x$ satisfies $y'' = -y$, or better to add $z = \cos x$, then $y' = z$ and $z' = -y$ with $y(0) = 0$, $z(0) = 1$. Let $y = \sum_{k=1}^{\infty} a_k x^k$ and $z = 1 + \sum_{k=1}^{\infty} b_k x^k$.

Then $\sum_{k=1}^{\infty} k a_k x^{k-1} = 1 + \sum_{k=1}^{\infty} b_k x^k$ and $\sum_{k=1}^{\infty} k b_k x^{k-1} = \sum_{k=1}^{\infty} a_k x^k$,

JAMES MADISON UNIVERSITY

Some Simple Examples

- If $y = e^x$ then $y' = e^x$ so $y' = y$ with $y(0) = 1$. Let $y = 1 + \sum_{k=1}^{\infty} a_k x^k$.

Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$.
Equate powers of x to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.

- $y = \sin x$ satisfies $y'' = -y$, or better to add $z = \cos x$, then $y' = z$ and $z' = -y$ with $y(0) = 0$, $z(0) = 1$. Let $y = \sum_{k=1}^{\infty} a_k x^k$ and $z = 1 + \sum_{k=1}^{\infty} b_k x^k$.

Then $\sum_{k=1}^{\infty} k a_k x^{k-1} = 1 + \sum_{k=1}^{\infty} b_k x^k$ and $\sum_{k=1}^{\infty} k b_k x^{k-1} = \sum_{k=1}^{\infty} a_k x^k$,
So $a_i = b_{i-1}/i$ and $b_i = -a_{i-1}/i$. Thus, $a_1 = 1$, $b_1 = 0$, $a_2 = 0$, $b_2 = -1/2!$, $a_3 = -1/3!$, $b_3 = 0$, $a_4 = 0$, $b_4 = 1/4!$ and so on.

JAMES MADISON UNIVERSITY

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

JAMES MADISON UNIVERSITY.

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

Cauchy product: $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} c_i x_i$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

Cauchy product: $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} c_i x_i$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2$.

JAMES MADISON UNIVERSITY

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

Cauchy product: $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} c_i x_i$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2$.

So $a_1 = 1$,

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

Cauchy product: $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} c_i x_i$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2$.

So $a_1 = 1$, $2a_2 = a_1 + a_1$ or $a_2 = 1$,

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

Cauchy product: $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} c_i x_i$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2$.

So $a_1 = 1,\ \ 2a_2 = a_1 + a_1$ or $a_2 = 1,\ \ 3a_3 = a_2 + a_1^2 + a_2$ or $a_3 = 1$,

JAMES MADISON UNIVERSITY

Slightly Less Simple Example

If $y = 1/(1-x)$ then $y' = 1/(1-x)^2$ or $y' = y^2$ with $y(0) = 1$.

Cauchy product: $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} c_i x_i$ where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = (1 + a_1x + a_2x^2 + a_3x^3 + \cdots)^2$.
So $a_1 = 1$, $2a_2 = a_1 + a_1$ or $a_2 = 1$, $3a_3 = a_2 + a_1^2 + a_2$ or $a_3 = 1$,
$4a_4 = a_3 + a_1a_2 + a_2a_1 + a_3$ or $a_4 = 1$ and so on.

Or, $(x-1)y = 1$, multiply out the left hand side and equate coefficients to get the required answer.

A Difficult Example

If $y = \tan x$ then $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ with $y(0) = 0$.

JAMES MADISON UNIVERSITY

A Difficult Example

If $y = \tan x$ then $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ with $y(0) = 0$.

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)^2.$$

A Difficult Example

If $y = \tan x$ then $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ with $y(0) = 0$.

$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots = 1 + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)^2$.

So $a_1 = 1$, $2a_2 = 0$ or $a_2 = 0$, $3a_3 = a_1^2$ or $a_3 = 1/3$, $4a_4 = 2a_1a_2$ or $a_4 = 0$, $5a_5 = 2a_1a_3 + a_2^2$ or $a_5 = 2/15$, $6a_6 = 2a_1a_4 + 2a_2a_3$ or $a_6 = 0$, $7a_7 = 2a_1a_5 + 2a_2a_4 + a_3^2$ or $a_7 = 17/315$ and so on.

A Difficult Example

If $y = \tan x$ then $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ with $y(0) = 0$.

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots = 1 + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)^2.$$

So $a_1 = 1$, $2a_2 = 0$ or $a_2 = 0$, $3a_3 = a_1^2$ or $a_3 = 1/3$, $4a_4 = 2a_1a_2$ or $a_4 = 0$, $5a_5 = 2a_1a_3 + a_2^2$ or $a_5 = 2/15$, $6a_6 = 2a_1a_4 + 2a_2a_3$ or $a_6 = 0$, $7a_7 = 2a_1a_5 + 2a_2a_4 + a_3^2$ or $a_7 = 17/315$ and so on.

For even n, $a_n = \dfrac{2}{n} \displaystyle\sum_{i=1}^{n/2-1} a_i a_{n-i-1}$, and since by induction $a_i = 0$ for every even $i < n$, and one of a_i, a_{n-i-1} is even, $a_n = 0$.

JAMES MADISON UNIVERSITY

A Difficult Example

If $y = \tan x$ then $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ with $y(0) = 0$.

$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots = 1 + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)^2$.

So $a_1 = 1$, $2a_2 = 0$ or $a_2 = 0$, $3a_3 = a_1^2$ or $a_3 = 1/3$, $4a_4 = 2a_1a_2$ or $a_4 = 0$, $5a_5 = 2a_1a_3 + a_2^2$ or $a_5 = 2/15$, $6a_6 = 2a_1a_4 + 2a_2a_3$ or $a_6 = 0$, $7a_7 = 2a_1a_5 + 2a_2a_4 + a_3^2$ or $a_7 = 17/315$ and so on.

For even n, $a_n = \frac{2}{n}\sum_{i=1}^{n/2-1} a_i a_{n-i-1}$, and since by induction $a_i = 0$ for every even $i < n$, and one of a_i, a_{n-i-1} is even, $a_n = 0$.

For odd n, $a_n = \frac{1}{n}\left(\sum_{i=1}^{(n-3)/2} 2a_i a_{n-i-1} + a_{(n-1)/2}^2\right)$. Since $a_n = 0$ for even n, there are two cases:

JAMES MADISON UNIVERSITY

Tan and Bernoulli Numbers

If $n \pmod{4} \equiv 1$ then $a_n = \dfrac{2}{n} \displaystyle\sum_{i=1}^{(n-1)/4} a_{2i-1} a_{n-2i}$,

and if $n \pmod{4} \equiv 3$ then $a_n = \dfrac{1}{n} \left(\displaystyle\sum_{i=1}^{(n-3)/4} 2a_{2i-1} a_{n-2i} + a_{(n-1)/2}^2 \right)$.

JAMES MADISON UNIVERSITY

Tan and Bernoulli Numbers

If $n \pmod 4 \equiv 1$ then $a_n = \frac{2}{n} \sum_{i=1}^{(n-1)/4} a_{2i-1} a_{n-2i}$,

and if $n \pmod 4 \equiv 3$ then $a_n = \frac{1}{n} \left(\sum_{i=1}^{(n-3)/4} 2a_{2i-1} a_{n-2i} + a_{(n-1)/2}^2 \right)$.

But Bernoulli numbers satisfy $\tan x = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} 4^i (4^i - 1) B_{2i}}{(2i)!} x^{2i-1}$,

Tan and Bernoulli Numbers

If $n \pmod 4 \equiv 1$ then $a_n = \frac{2}{n} \sum_{i=1}^{(n-1)/4} a_{2i-1}a_{n-2i}$,

and if $n \pmod 4 \equiv 3$ then $a_n = \frac{1}{n}\left(\sum_{i=1}^{(n-3)/4} 2a_{2i-1}a_{n-2i} + a_{(n-1)/2}^2\right)$.

But Bernoulli numbers satisfy $\tan x = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}4^i(4^i-1)B_{2i}}{(2i)!}x^{2i-1}$, so these recurrences give a new set of formulas for finding Bernoulli numbers.

JAMES MADISON UNIVERSITY

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

Let $c = \sin a$ and $d = \cos a$.

JAMES MADISON UNIVERSITY

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

Let $c = \sin a$ and $d = \cos a$.

Then $b' = c$,

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

Let $c = \sin a$ and $d = \cos a$.

Then $b' = c$,

$c' = a' \cos a$ or $c' = bd$, with $c(0) = \sin y_0$,

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

Let $c = \sin a$ and $d = \cos a$.

Then $b' = c$,

$c' = a' \cos a$ or $c' = bd$, with $c(0) = \sin y_0$,

and $d' = -a' \sin a$ or $d' = -bc$ with $d(0) = \cos y_0$.

JAMES MADISON UNIVERSITY

$y'' = \sin y$

Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

Let $c = \sin a$ and $d = \cos a$.

Then $b' = c$,

$c' = a' \cos a$ or $c' = bd$, with $c(0) = \sin y_0$,

and $d' = -a' \sin a$ or $d' = -bc$ with $d(0) = \cos y_0$.

Taylor series for these four odes are easily found by hand, and symbolic packages deal with them instantly.

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \dfrac{1}{a}$, then $b' = -\dfrac{1}{a^2}a' = -\dfrac{1}{a}.\dfrac{a'}{a}$.

JAMES MADISON UNIVERSITY

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \dfrac{1}{a}$, then $b' = -\dfrac{1}{a^2}a' = -\dfrac{1}{a}.\dfrac{a'}{a}$. The initial condition is $b(0) = \dfrac{1}{a(0)}$.

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \frac{1}{a}$, then $b' = -\frac{1}{a^2}a' = -\frac{1}{a}.\frac{a'}{a}$. The initial condition is $b(0) = \frac{1}{a(0)}$.

Let $c = \frac{a'}{a}$, then $b' = -b \cdot c$

JAMES MADISON UNIVERSITY

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \frac{1}{a}$, then $b' = -\frac{1}{a^2}a' = -\frac{1}{a}.\frac{a'}{a}$. The initial condition is $b(0) = \frac{1}{a(0)}$.

Let $c = \frac{a'}{a}$, then $b' = -b \cdot c$ and $c' = \frac{a \cdot a'' - a'^2}{a^2} = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = ca'' - c^2$.

JAMES MADISON UNIVERSITY

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \frac{1}{a}$, then $b' = -\frac{1}{a^2}a' = -\frac{1}{a}.\frac{a'}{a}$. The initial condition is $b(0) = \frac{1}{a(0)}$.

Let $c = \frac{a'}{a}$, then $b' = -b \cdot c$ and $c' = \frac{a \cdot a'' - a'^2}{a^2} = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = ca'' - c^2$.

The initial condition is $c(0) = \frac{a'(0)}{a(0)}$.

JAMES MADISON UNIVERSITY

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \frac{1}{a}$, then $b' = -\frac{1}{a^2}a' = -\frac{1}{a}.\frac{a'}{a}$. The initial condition is $b(0) = \frac{1}{a(0)}$.

Let $c = \frac{a'}{a}$, then $b' = -b \cdot c$ and $c' = \frac{a \cdot a'' - a'^2}{a^2} = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = ca'' - c^2$.
The initial condition is $c(0) = \frac{a'(0)}{a(0)}$.

Formal power series substitution immediately gives the Taylor series for b and c term by term using three Cauchy products.

JAMES MADISON UNIVERSITY

Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$.

Let $b = \dfrac{1}{a}$, then $b' = -\dfrac{1}{a^2}a' = -\dfrac{1}{a}.\dfrac{a'}{a}$. The initial condition is $b(0) = \dfrac{1}{a(0)}$.

Let $c = \dfrac{a'}{a}$, then $b' = -b \cdot c$ and $c' = \dfrac{a \cdot a'' - a'^2}{a^2} = \dfrac{a''}{a} - \left(\dfrac{a'}{a}\right)^2 = ca'' - c^2$.
The initial condition is $c(0) = \dfrac{a'(0)}{a(0)}$.

Formal power series substitution immediately gives the Taylor series for b and c term by term using three Cauchy products.

Alternatively, if $b = 1/a$ then $a \cdot b = 1$. With a single Cauchy product and equating coefficients, each b_n can be found as a function of $a_0, a_1, \ldots, a_n$ and (known) $b_0, b_1, \ldots, b_{n-1}$.

JAMES MADISON UNIVERSITY

Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real r.

Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real r.

Similarly to before, define $b = a^r$ where a has a known Taylor series. Then let $c = 1/a$ and $d = a'/a$.

Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real r.

Similarly to before, define $b = a^r$ where a has a known Taylor series. Then let $c = 1/a$ and $d = a'/a$.

Then $b' = ra^{r-1}a' = ra^r \cdot \dfrac{a'}{a} = rb \cdot d$ with $b(0) = (a(0))^r$.

Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real r.

Similarly to before, define $b = a^r$ where a has a known Taylor series. Then let $c = 1/a$ and $d = a'/a$.

Then $b' = ra^{r-1}a' = ra^r \cdot \frac{a'}{a} = rb \cdot d$ with $b(0) = (a(0))^r$.

$c' = -\frac{a'}{a^2} - -\frac{1}{a} \cdot \frac{a'}{a} = -c \cdot d$ with $c(0) = \frac{1}{a(0)}$.

JAMES MADISON UNIVERSITY

Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real r.

Similarly to before, define $b = a^r$ where a has a known Taylor series. Then let $c = 1/a$ and $d = a'/a$.

Then $b' = ra^{r-1}a' = ra^r \cdot \dfrac{a'}{a} = rb \cdot d$ with $b(0) = (a(0))^r$.

$c' = -\dfrac{a'}{a^2} - -\dfrac{1}{a} \cdot \dfrac{a'}{a} = -c \cdot d$ with $c(0) = \dfrac{1}{a(0)}$.

$d' = \dfrac{a \cdot a'' - a'^2}{a^2} = \dfrac{a''}{a} - \left(\dfrac{a'}{a}\right)^2 = c \cdot a'' - d^2$ with $d(0) = \dfrac{a'(0)}{a(0)}$.

JAMES MADISON UNIVERSITY

Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real r.

Similarly to before, define $b = a^r$ where a has a known Taylor series. Then let $c = 1/a$ and $d = a'/a$.

Then $b' = ra^{r-1}a' = ra^r \cdot \dfrac{a'}{a} = rb \cdot d$ with $b(0) = (a(0))^r$.

$c' = -\dfrac{a'}{a^2} - -\dfrac{1}{a} \cdot \dfrac{a'}{a} = -c \cdot d$ with $c(0) = \dfrac{1}{a(0)}$.

$d' = \dfrac{a \cdot a'' - a'^2}{a^2} = \dfrac{a''}{a} - \left(\dfrac{a'}{a}\right)^2 = c \cdot a'' - d^2$ with $d(0) = \dfrac{a'(0)}{a(0)}$.

Or (Warne) from $b' = ra^{r-1}a'$, $ab' = rba'$, and only two Cauchy products lead to $b_n = \dfrac{1}{na_0} \displaystyle\sum_{k=1}^{n} ((r+1)k - n)a_k b_{n-k}$.

JAMES MADISON UNIVERSITY

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$.

JAMES MADISON UNIVERSITY

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$.

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

JAMES MADISON UNIVERSITY

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \dfrac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$.

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$,

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2 p_n$.

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2 p_n$.

$p_n' = f'''(g)g'$ or $p_n' = p_{n-1}y$ with $p_{n-1} = f'''(g)$.

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2 p_n$.

$p_n' = f'''(g)g'$ or $p_n' = p_{n-1}y$ with $p_{n-1} = f'''(g)$.

$p_{n-1}' = f^{(4)}(g)g'$ or $p_{n-1}' = p_{n-2}y$ with $p_{n-2} = f^{(4)}(g)$.

JAMES MADISON UNIVERSITY

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2 p_n$.

$p_n' = f'''(g)g'$ or $p_n' = p_{n-1}y$ with $p_{n-1} = f'''(g)$.

$p_{n-1}' = f^{(4)}(g)g'$ or $p_{n-1}' = p_{n-2}y$ with $p_{n-2} = f^{(4)}(g)$.

and so on until $p_1' = f^{(n+2)}(g)g'$

JAMES MADISON UNIVERSITY

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \dfrac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2 p_n$.

$p_n' = f'''(g)g'$ or $p_n' = p_{n-1}y$ with $p_{n-1} = f'''(g)$.

$p_{n-1}' = f^{(4)}(g)g'$ or $p_{n-1}' = p_{n-2}y$ with $p_{n-2} = f^{(4)}(g)$.

and so on until $p_1' = f^{(n+2)}(g)g'$ or $p_1' = (n+2)!a_{n+2}y$.

Inverses of Polynomials (Sochacki & Parker)

Let $f(t) = \sum_{i=0}^{n+2} a_i t_i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2 p_n$.

$p_n' = f'''(g)g'$ or $p_n' = p_{n-1}y$ with $p_{n-1} = f'''(g)$.

$p_{n-1}' = f^{(4)}(g)g'$ or $p_{n-1}' = p_{n-2}y$ with $p_{n-2} = f^{(4)}(g)$.

and so on until $p_1' = f^{(n+2)}(g)g'$ or $p_1' = (n+2)!a_{n+2}y$.

Finding the Taylor series for the (red) system of odes gives $y = g'(t)$, and an integration gives the required inverse. Simple (if tedious!)