Taylor Series Without High Order Differentiation

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Outline

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- A new Bernoulli number algorithm.
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- Straightforward Examples.
- A new Bernoulli number algorithm.
- Reciprocals and general powers of functions.
- The inverse of a polynomial.
Taylor Series

Assuming $f$ is sufficiently differentiable,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$
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f^{(4)}(x) = \sin \sqrt{x^2 - 1} \left(\frac{x^4}{(x^2-1)^{5/2}} - \frac{15x^4}{(x^2-1)^{3/2}} + \frac{18x^2}{(x^2-1)^{7/2}} - \frac{3}{(x^2-1)^{3/2}}\right) + \cos \sqrt{x^2 - 1} \left(\frac{6x^4}{(x^2-1)^{5/2}} - \frac{6x^2}{(x^2-1)^{3/2}} - \frac{15x^4}{(x^2-1)^{7/2}} + \frac{18x^2}{(x^2-1)^{5/2}} - \frac{3}{(x^2-1)^{3/2}}\right),
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$$+ \cos \sqrt{x^2 - 1} \left( \frac{6x^4}{(x^2 - 1)^{5/2}} - \frac{6x^2}{(x^2 - 1)^{3/2}} - \frac{15x^4}{(x^2 - 1)^{7/2}} + \frac{18x^2}{(x^2 - 1)^{5/2}} - \frac{3}{(x^2 - 1)^{3/2}} \right),$$

and the tenth derivative has 45 terms.
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$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

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and the tenth derivative has 45 terms. Also Maple “hangs” when trying to solve $y'' = \sin(y)$ using series after about 8 terms.
An Ode Solution Technique

A first course in differential equations introduces a power series substitution method for second order linear differential equations of the form \( p(x)y'' + q(x)y' + r(x)y = f(x) \), as long as \( p, q, r, f \) are sufficiently simple.
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Almost never seen again, especially when an ode is nonlinear. But, there is no reason why it can’t be applied to these odes with some alterations, particularly those whose solution is a function whose Taylor series we want...
Solution Approach

The power series substitution method works because the left and right hand sides can be expanded as power series, and coefficients of successive powers of $x$ equated.
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Add additional variables and their derivatives in such a way that we get a system of first order equations of the form $Y' = F(Y)$, where each right hand side is polynomial in the variables and $x$. 
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We call the class of functions that can be formulated in this way Picardable.
Why Picardable?

Ed Parker and Jim Sochacki showed that the system $Y' = F(Y)$ where the right hands sides are polynomial in the variables could be solved as Taylor series using Picard’s method around a point where the functions are analytic.
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Later, Paul Warne showed that the same Taylor series could be obtained by formal power substitution, and David Carothers showed that every pp system could be rewritten so that only quadratic polynomial terms are required, so only Cauchy products of power series are needed.
Consider \( y = \frac{e^x - 1}{x} \), which is analytic at zero.
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But consider $xy' = 1 - y + z$ and $z' = z + 1$ with $y(0) = 1$ and $z(0) = 0$. 
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Conjecture: Picardable equals those analytic functions whose power series hold a finite amount of information.
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Then $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. 
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Equate powers of $x$ to get $a_1 = 1$, $2a_2 = a_1$ or $a_2 = 1/2$, $3a_3 = a_2$ or $a_3 = 1/3!$, $4a_4 = a_3$ or $a_4 = 1/4!$, and so on.
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- $y = \sin x$ satisfies $y'' = -y$, or better to add $z = \cos x$, then $y' = z$ and $z' = -y$ with $y(0) = 0$, $z(0) = 1$. 
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  Then $\sum_{k=1}^{\infty} k a_k x^{k-1} = 1 + \sum_{k=1}^{\infty} b_k x^k$ and $\sum_{k=1}^{\infty} k b_k x^{k-1} = \sum_{k=1}^{\infty} a_k x^k$. 
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• \( y = \sin x \) satisfies \( y'' = -y \), or better to add \( z = \cos x \), then \( y' = z \) and \( z' = -y \) with \( y(0) = 0, \ z(0) = 1 \). Let \( y = \sum_{k=1}^{\infty} a_k x^k \) and \( z = 1 + \sum_{k=1}^{\infty} b_k x^k \).

Then \( \sum_{k=1}^{\infty} ka_k x^{k-1} = 1 + \sum_{k=1}^{\infty} b_k x^k \) and \( \sum_{k=1}^{\infty} kb_k x^{k-1} = \sum_{k=1}^{\infty} a_k x^k \).

So \( a_i = b_{i-1}/i \) and \( b_i = -a_{i-1}/i \). Thus, \( a_1 = 1, b_1 = 0, a_2 = 0, b_2 = -1/2! \), \( a_3 = -1/3! \), \( b_3 = 0, a_4 = 0, b_4 = 1/4! \) and so on.
Slightly Less Simple Example

If \( y = 1/(1 - x) \) then \( y' = 1/(1 - x)^2 \) or \( y' = y^2 \) with \( y(0) = 1 \).
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Cauchy product: \( \left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{i=0}^{\infty} b_i x^i \right) = \sum_{i=0}^{\infty} c_i x_i \) where \( c_i = \sum_{j=0}^{i} a_j b_{i-j} \).
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\[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2.\]
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So \( a_1 = 1 \),
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So \( a_1 = 1 \), \( 2a_2 = a_1 + a_1 \) or \( a_2 = 1 \),
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\[ a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2. \]

So \( a_1 = 1, \ 2a_2 = a_1 + a_1 \) or \( a_2 = 1, \ 3a_3 = a_2 + a_1^2 + a_2 \) or \( a_3 = 1 \),
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\[ a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = (1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2. \]

So \( a_1 = 1, \ 2a_2 = a_1 + a_1 \) or \( a_2 = 1, \ 3a_3 = a_2 + a_1^2 + a_2 \) or \( a_3 = 1, \)
\( 4a_4 = a_3 + a_1 a_2 + a_2 a_1 + a_3 \) or \( a_4 = 1 \) and so on.

Or, \((x - 1)y = 1\), multiply out the left hand side and equate coefficients to get the required answer.
A Difficult Example

If $y = \tan x$ then $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ with $y(0) = 0$. 
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\[ a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)^2. \]
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$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots = 1 + (a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)^2.$$ 

So $a_1 = 1$, $2a_2 = 0$ or $a_2 = 0$, $3a_3 = a_1^2$ or $a_3 = 1/3$, $4a_4 = 2a_1a_2$ or $a_4 = 0$, $5a_5 = 2a_1a_3 + a_2^2$ or $a_5 = 2/15$, $6a_6 = 2a_1a_4 + 2a_2a_3$ or $a_6 = 0$, $7a_7 = 2a_1a_5 + 2a_2a_4 + a_3^2$ or $a_7 = 17/315$ and so on.
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$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)^2.$$  

So $a_1 = 1$, $2a_2 = 0$ or $a_2 = 0$, $3a_3 = a_1^2$ or $a_3 = 1/3$, $4a_4 = 2a_1 a_2$ or $a_4 = 0$, $5a_5 = 2a_1 a_3 + a_2^2$ or $a_5 = 2/15$, $6a_6 = 2a_1 a_4 + 2a_2 a_3$ or $a_6 = 0$, $7a_7 = 2a_1 a_5 + 2a_2 a_4 + a_3^2$ or $a_7 = 17/315$ and so on.

For even $n$, $a_n = \frac{2}{n} \sum_{i=1}^{n/2-1} a_i a_{n-i-1}$, and since by induction $a_i = 0$ for every even $i < n$, and one of $a_i$, $a_{n-i-1}$ is even, $a_n = 0$. 
If \( y = \tan x \) then \( y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \) with \( y(0) = 0 \).

\[
a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots = 1 + (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)^2.
\]

So \( a_1 = 1, 2a_2 = 0 \) or \( a_2 = 0, 3a_3 = a_1^2 \) or \( a_3 = 1/3, 4a_4 = 2a_1 a_2 \) or \( a_4 = 0, 5a_5 = 2a_1 a_3 + a_2^2 \) or \( a_5 = 2/15, 6a_6 = 2a_1 a_4 + 2a_2 a_3 \) or \( a_6 = 0, 7a_7 = 2a_1 a_5 + 2a_2 a_4 + a_3^2 \) or \( a_7 = 17/315 \) and so on.

For even \( n \), \( a_n = \frac{2}{n} \sum_{i=1}^{n/2-1} a_i a_{n-i-1} \), and since by induction \( a_i = 0 \) for every even \( i < n \), and one of \( a_i, a_{n-i-1} \) is even, \( a_n = 0 \).

For odd \( n \), \( a_n = \frac{1}{n} \left( \sum_{i=1}^{(n-3)/2} 2a_i a_{n-i-1} + a_{(n-1)/2}^2 \right) \). Since \( a_n = 0 \) for even \( n \), there are two cases:
Tan and Bernoulli Numbers

If \( n \pmod{4} \equiv 1 \) then
\[
a_n = \frac{2}{n} \sum_{i=1}^{(n-1)/4} a_{2i-1}a_{n-2i},
\]

and if \( n \pmod{4} \equiv 3 \) then
\[
a_n = \frac{1}{n} \left( \sum_{i=1}^{(n-3)/4} 2a_{2i-1}a_{n-2i} + a^2(n-1)/2 \right).
\]
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But Bernoulli numbers satisfy \( \tan x = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} 4^i (4^i - 1) B_{2i}}{(2i)!} x^{2i-1} \),
Tan and Bernoulli Numbers

If \( n \pmod{4} \equiv 1 \) then \( a_n = \frac{2}{n} \sum_{i=1}^{(n-1)/4} a_{2i-1}a_{n-2i} \),

and if \( n \pmod{4} \equiv 3 \) then \( a_n = \frac{1}{n} \left( \sum_{i=1}^{(n-3)/4} 2a_{2i-1}a_{n-2i} + a_2^{(n-1)/2} \right) \).

But Bernoulli numbers satisfy \( \tan x = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}4^i(4^i - 1)B_{2i}}{(2i)!} x^{2i-1} \), so these recurrences give a new set of formulas for finding Bernoulli numbers.
Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$. 
Using this approach, consider \( y'' = \sin y \) with \( y(0) = y_0, \ y'(0) = y_1 \).

Let \( a = y \) and \( b = y' \), then \( a' = b \) and \( b' = \sin a \) with \( a(0) = y_0, \ b(0) = y_1 \).
Using this approach, consider $y'' = \sin y$ with $y(0) = y_0$, $y'(0) = y_1$.

Let $a = y$ and $b = y'$, then $a' = b$ and $b' = \sin a$ with $a(0) = y_0$, $b(0) = y_1$.

Let $c = \sin a$ and $d = \cos a$. 
Using this approach, consider \( y'' = \sin y \) with \( y(0) = y_0, y'(0) = y_1 \).

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Then \( b' = c \),
\[ y'' = \sin y \]

Using this approach, consider \( y'' = \sin y \) with \( y(0) = y_0, y'(0) = y_1 \).

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Let \( c = \sin a \) and \( d = \cos a \).

Then \( b' = c \),

\[ c' = a' \cos a \text{ or } c' = bd, \text{ with } c(0) = \sin y_0, \]
Using this approach, consider \( y'' = \sin y \) with \( y(0) = y_0, \ y'(0) = y_1 \).

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\[
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\]

with \( c(0) = \sin y_0 \),

and \( d' = -a' \sin a \) or \( d' = -bc \) with \( d(0) = \cos y_0 \).
Using this approach, consider \( y'' = \sin y \) with \( y(0) = y_0, \ y'(0) = y_1 \).

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Taylor series for these four odes are easily found by hand, and symbolic packages deal with them instantly.
Reciprocal of Analytic Functions

Let $a(x)$ be a function with a known Taylor series (perhaps from a system of polynomial odes). We wish to find the Taylor series of $1/a(x)$. 
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Let \( c = \frac{a'}{a} \), then \( b' = -b \cdot c \) and \( c' = \frac{a \cdot a'' - a'^2}{a^2} = \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 = ca'' - c^2 \).
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Formal power series substitution immediately gives the Taylor series for \( b \) and \( c \) term by term using three Cauchy products.
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Formal power series substitution immediately gives the Taylor series for \( b \) and \( c \) term by term using three Cauchy products.

Alternatively, if \( b = 1/a \) then \( a \cdot b = 1 \). With a single Cauchy product and equating coefficients, each \( b_n \) can be found as a function of \( a_0, a_1, \ldots, a_n \) and (known) \( b_0, b_1, \ldots, b_{n-1} \).
Powers of Analytic Functions

Let $a(x)$ be a function whose Taylor series is known (perhaps from a system of polynomial odes). We wish to find $a(x)^r$ for any real $r$. 
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Similarly to before, define $b = a^r$ where $a$ has a known Taylor series. Then let $c = 1/a$ and $d = a'/a$. 
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Then $b' = ra^{r-1}a' = ra^r \cdot \frac{a'}{a} = rb \cdot d$ with $b(0) = (a(0))^r$. 
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Then $b' = r a^{r-1} a' = r a^r \cdot \frac{a'}{a} = r b \cdot d$ with $b(0) = (a(0))^r$.

$$c' = -\frac{a'}{a^2} - \frac{1}{a} \cdot \frac{a'}{a} = -c \cdot d$$ with $c(0) = \frac{1}{a(0)}$. 
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\]

\[
d' = \frac{a \cdot a'' - a'^2}{a^2} = \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 = c \cdot a'' - d^2 \text{ with } d(0) = \frac{a'(0)}{a(0)}.
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Or (Warne) from $b' = ra^{r-1}a'$, $ab' = rba'$, and only two Cauchy products lead to $b_n = \frac{1}{na_0} \sum_{k=1}^{n} ((r + 1)k - n)a_k b_{n-k}$. 
Inverses of Polynomials (Sochacki & Parker)

Let \( f(t) = \sum_{i=0}^{n+2} a_i t^i \) and \( f(g(t)) = t \), so \( g = f^{-1} \).
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Let \( f(t) = \sum_{i=0}^{n+2} a_i t^i \) and \( f(g(t)) = t \), so \( g = f^{-1} \). Then \( f(g) = t \), or

\[ f'(g) g' = 1. \]
Let $f(t) = \sum_{i=0}^{n+2} a_i t^i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$. 
Let \( f(t) = \sum_{i=0}^{n+2} a_i t^i \) and \( f(g(t)) = t \), so \( g = f^{-1} \). Then \( f(g) = t \), or \( f''(g)g' = 1 \). Letting \( y = g'(t) \), then \( y = 1/f'(g) \).

So \( y' = \frac{-1}{(f'(g))^2} f'''(g)g' = -y^2 f''(g)y \).
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Let \( f(t) = \sum_{i=0}^{n+2} a_i t^i \) and \( f(g(t)) = t \), so \( g = f^{-1} \). Then \( f(g) = t \), or \( f'(g)g' = 1 \). Letting \( y = g'(t) \), then \( y = 1/f'(g) \).

So \( y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y \). Let \( x = y^2 \) and \( p_n = f''(g) \), then \( y' = -xp_n y \), and \( x' = 2yy' = 2y(-xp_n y) \) or \( x' = -2x^2p_n \).
Let \( f(t) = \sum_{i=0}^{n+2} a_i t^i \) and \( f(g(t)) = t \), so \( g = f^{-1} \). Then \( f(g) = t \), or \( f'(g)g' = 1 \). Letting \( y = g'(t) \), then \( y = 1/f'(g) \).

So \( y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y \). Let \( x = y^2 \) and \( p_n = f'''(g) \), then \( y' = -xp_n y \), and \( x' = 2yy' = 2y(-xp_n y) \) or \( x' = -2x^2 p_n \).

\( p'_n = f'''(g)g' \) or \( p'_n = p_{n-1}y \) with \( p_{n-1} = f'''(g) \).
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Let $f(t) = \sum_{i=0}^{n+2} a_i t^i$ and $f(g(t)) = t$, so $g = f^{-1}$. Then $f(g) = t$, or $f'(g)g' = 1$. Letting $y = g'(t)$, then $y = 1/f'(g)$.

So $y' = -\frac{1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2p_n$.

$p'_n = f'''(g)g'$ or $p'_n = p_{n-1} y$ with $p_{n-1} = f'''(g)$.

$p'_{n-1} = f^{(4)}(g)g'$ or $p'_{n-1} = p_{n-2} y$ with $p_{n-2} = f^{(4)}(g)$.
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Let \( f(t) = \sum_{i=0}^{n+2} a_i t^i \) and \( f(g(t)) = t \), so \( g = f^{-1} \). Then \( f(g) = t \), or \( f'(g)g' = 1 \). Letting \( y = g'(t) \), then \( y = 1/f'(g) \).

So \( y' = \frac{-1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y \). Let \( x = y^2 \) and \( p_n = f''(g) \), then \( y' = -xp_ny \), and \( x' = 2yy' = 2y(-xp_ny) \) or \( x' = -2x^2p_n \).

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and so on until \( p'_1 = f^{(n+2)}(g)g' \).
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So \( y' = -\frac{1}{(f'(g))^2} f'''(g)g' = -y^2 f'''(g)y \). Let \( x = y^2 \) and \( p_n = f'''(g) \), then \( y' = -xp_n y \), and \( x' = 2yy' = 2y(-xp_n y) \) or \( x' = -2x^2 p_n \).

\( p'_n = f'''(g)g' \) or \( p'_n = p_{n-1}y \) with \( p_{n-1} = f'''(g) \).

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So $y' = -\frac{1}{(f'(g))^2} f''(g)g' = -y^2 f''(g)y$. Let $x = y^2$ and $p_n = f''(g)$, then $y' = -xp_n y$, and $x' = 2yy' = 2y(-xp_n y)$ or $x' = -2x^2p_n$.

$p_n' = f'''(g)g'$ or $p_n' = p_{n-1}y$ with $p_{n-1} = f'''(g)$.

$p_{n-1}' = f^{(4)}(g)g'$ or $p_{n-1}' = p_{n-2}y$ with $p_{n-2} = f^{(4)}(g)$.

and so on until $p_1' = f^{(n+2)}(g)g'$ or $p_1' = (n + 2)!a_{n+2}y$.

Finding the Taylor series for the (red) system of odes gives $y = g'(t)$, and an integration gives the required inverse. Simple (if tedious!)