Numerically evaluating oscillating infinite integrals

and a failed (of course) approach to the Riemann Hypothesis

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September 15 2014

Outline

- Why infinite oscillatory integrals?
- Techniques for oscillatory integrals.
- Techniques for multiple period oscillations.
- What is the Riemann hypothesis.
- X-ray plots and a conjecture.
- The (non)-applicability of oscillatory integration theory.

Thanks to Howard Stone (Princeton), Jim Hill (Wollongong)

The Electrified Disk

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2}=0, \quad \begin{cases} V=V_0, \ 0\le r<1, \ z=0 \\ \dfrac{\partial V}{\partial z}=0, \ r>1, \ z=0 \\ V\to 0 \text{ as } \sqrt{r^2+z^2}\to\infty \end{cases}$$

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which has solution $\bar{V}=Ae^{-kz}$, or
$V(r,z)=\displaystyle\int_0^\infty A(k)e^{-kz}kJ_0(rk)\,dk.$

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Boundary conditions at $z=0$:

$$\begin{cases} \displaystyle\int_0^\infty kA(k)J_0(rk)\,dk=V_0 & 0\le r<1 \\ \displaystyle\int_0^\infty k^2A(k)J_0(rk)\,dk=0 & r>1 \end{cases}$$

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Dual Integral Equations for unknown $A(k)$

Tranter's Method

[C.J. Tranter, *Integral Equations in Mathematical Physics*, 1966.]

To solve
$$\begin{cases} \displaystyle\int_0^\infty G(k)f(k)J_\nu(rk)\,dk = g(r) \quad 0 \le r < 1 & (1) \\ \displaystyle\int_0^\infty f(k)J_\nu(rk)\,dk = 0 \quad r > 1 & (2) \end{cases}$$

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use the Weber-Schafheitlin discontinuous integral

$$\int_0^\infty k^{1-\beta} J_{2m+\nu+\beta}(k) J_\nu(rk)\, dk = \begin{cases} \dfrac{\Gamma(\nu+m+1) r^\nu (1-r^2)^{\beta-1}}{2^{\beta-1}\Gamma(\nu+1)\Gamma(m+\beta)} \\ \quad \times \mathcal{F}(\beta+\nu, \nu+1; r^2) \quad 0 \le r < 1 \\ 0 \quad r > 1 \end{cases}$$

where m is an integer ≥ 0, real $\beta > 0$, $\nu > -2-m$.

Tranter's Method (2)

Seek a solution $f(k) = k^{1-\beta} \sum_{m=0}^{\infty} a_m J_{2m+\nu+\beta}(k)$, which automatically satisfies (2).

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Substitute in (1), assume $g(r) = Ar^{\nu}$, and use orthogonality of Jacobi polynomials to give:

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} G(k) k^{1-2\beta} J_{2m+\nu+\beta}(k) J_{2n+\nu+\beta}(k)\, dk = \frac{A\Gamma(\nu+1)}{2^{\beta}\Gamma(\nu+\beta+1)} \delta_{0n}$$

for $n = 0, 1, 2, \ldots$.

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for $n = 0, 1, 2, \ldots$. Truncate and solve linear system of equations for a_m.

Choose β such that $k^{2-2\beta} G(k) - 1$ is as small as possible.

Applications of Dual Integral Equations

Tranter's method is useful for mixed boundary problems with disc or channel geometries. For example

- Motion of a circular disc in Stokes flow, broadside translation, edgewise translation, with and without boundaries, in a rotating viscous flow, oscillatory motion of a disc in unsteady Stokes flow.
- Capillary wave scattering.
- Fluid motion of monomolecular films in a channel flow.
- Flow of inviscid fluid around a disc in a pipe.
- Diffraction by elliptic and circular apertures in uniaxially anisotropic crystals.
- Various soil transportation models.

Green's Function Applications

The Green's functions for various problems are of the form $\int_0^\infty f(x)J_n(rx)\,dx$ or $\int_0^\infty f(x)J_a(\rho x)J_b(\tau x)\,dx$ for $n \in \mathbb{N}$, and $a, b \in \{0, 1\}$.

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- Particle motion in rotating viscous flows, and the Oseen equation.
- Magnetohydrodynamics.
- Antennas or scatterers embedded in planar multilayered media.
- Transversely isotropic piezoelectric multilayered half spaces.
- Isotropic elastic solid with a cylindrical borehole and a rigid plug.
- Scattering by cracks beneath fluid-solid interfaces.
- Response of a layered elastic half-space to surface loading.

Extrapolation for Summing Series

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- **Euler Transform:** $\displaystyle\sum_{i=0}^{\infty} u_i = \frac{1}{2}(u_0 + Mu_0 + M^2u_0 + \cdots)$ where $Mu_i = \frac{1}{2}(u_i + u_{i+1})$.

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- ϵ**-Algorithm:** (Implemented in QUADPACK/IMSL)

$$\epsilon_n^{(-1)} = 0, \quad \epsilon_n^{(0)} = I_n \quad \text{and} \quad \epsilon_n^{(p)} = \epsilon_{n+1}^{(p-2)} + \left[\epsilon_{n+1}^{(p-1)} - \epsilon_{n-1}^{(p-1)}\right]^{-1}$$

$\epsilon_n^{(2k)}$ is the kth order Shanks' transform of $\{I_n\}$

More Extrapolation

- **mW Transform:** (Sidi, 1988)

  To evaluate $\int_a^\infty g(x)\,dx$, form

$$F(x_s) = \int_a^{x_s} g(x)\,dx, \quad \psi(x_s) = \int_{x_s}^{x_{s+1}} g(x)\,dx,$$

$$M_{-1}^{(s)} = F(x_s)/\psi(x_s), \quad N_{-1}^{(s)} = 1/\psi(x_s),$$

$$M_p^{(s)} = \left(M_{p-1}^{(s)} - M_{p-1}^{(s+1)}\right) \Big/ \left(x_s^{-1} - x_{s+p+1}^{-1}\right)$$

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$$\Rightarrow \quad W_p^{(s)} = \frac{M_p^{(s)}}{N_p^{(s)}}, \quad s = 0, 1, \ldots, \quad p = 0, 1, \ldots$$

$W_p^{(0)}$ gives the best approximation to the integral.

Further Numerical Details

- Evaluate Bessel functions using IMSL routines or polynomial approximations from J.F. Hart, *Computer Approximations*, (1968).

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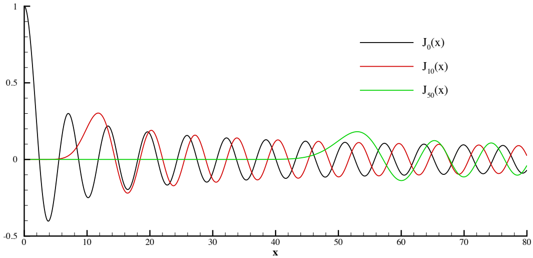
- Choosing interval endpoints as Bessel zeros (or midway between zeros, or approximate zeros, or offset zeros. . .).

Finding Bessel Zeros

- Bessel function zeros aren't evenly spaced



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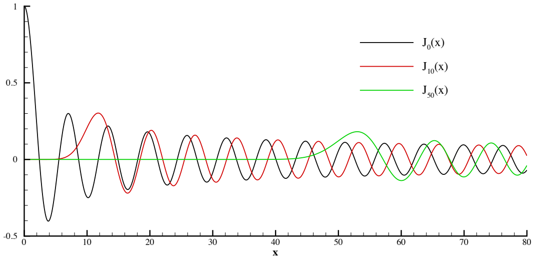

- Assume asymptotic zeros, place zeros π apart.
- Find zeros using Newton, $x_{i+1} = x_i - \dfrac{J_n(x_i)}{\frac{n}{x_i} J_n(x_i) - J_{n+1}(x_i)}$.

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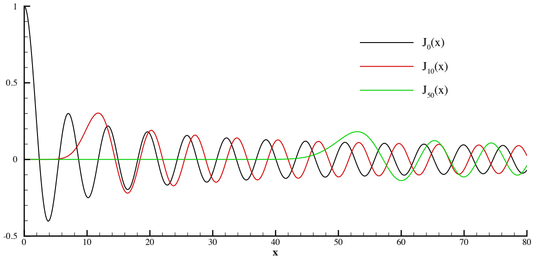

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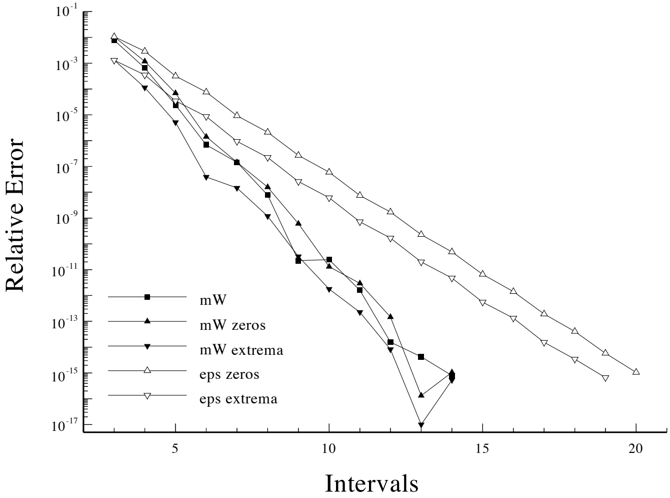
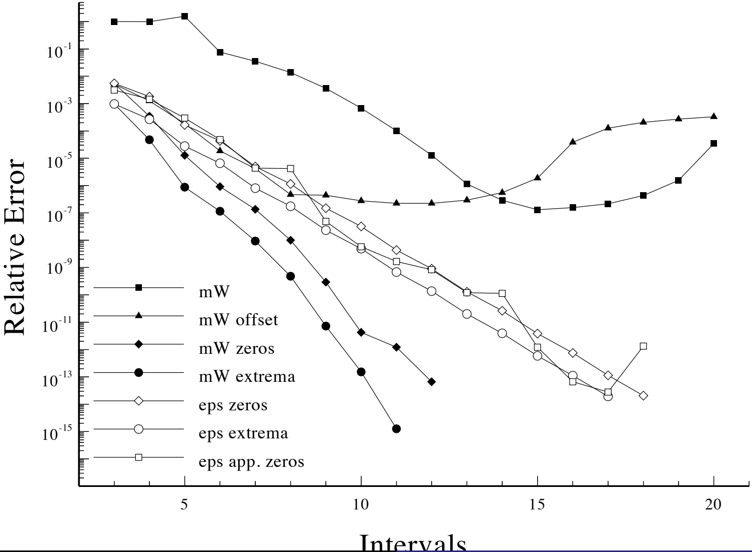
- For initial approximation to ith zero of $J_n(x)$ ($j_{n,i}$), use asymptotics for $j_{n,1}$, $j_{n,2}$ or simply $j_{n,i} \simeq j_{n,i-1} + (j_{n,i-1} - j_{n,i-2})$, $i \geq 3$.

$\int_0^\infty \frac{x}{1+x^2} J_0(x)\, dx$



$$\int_0^\infty \frac{x}{1+x^2} J_{10}(x)\,dx$$



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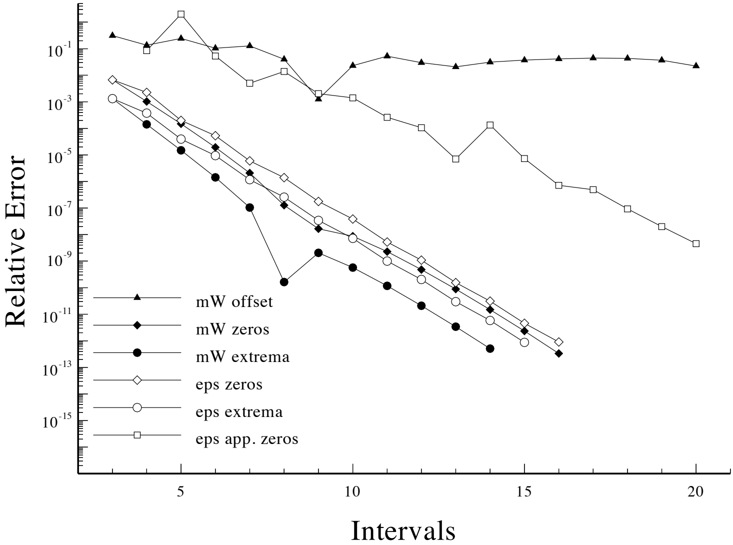

Summary of Extrapolation Methods

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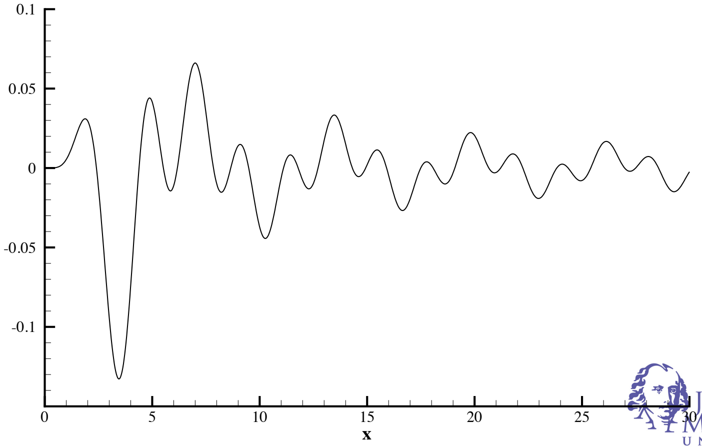
So,

If zeros are known
Then use mW transform
Else (zeros approximated) use ϵ-algorithm.

$I_{a,b,\rho,\tau} = \int_0^\infty f(x) J_a(\rho x) J_b(\tau x)\, dx$

With $f(x) = 1$, $a = 0$, $b = 5$, $\rho = 1$, $\tau = 3/2$:



The Transformation

Write

$$J_a(\rho x)J_b(\tau x) = h_1(x; a, b, \rho, \tau) + h_2(x; a, b, \rho, \tau),$$

$$h_1 = \frac{1}{2}\left\{J_a(\rho x)J_b(\tau x) - Y_a(\rho x)Y_b(\tau x)\right\}$$

$$h_2 = \frac{1}{2}\left\{J_a(\rho x)J_b(\tau x) + Y_a(\rho x)Y_b(\tau x)\right\}$$

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For large x,

$$h_1 \sim \frac{1}{\pi\sqrt{\rho\tau x}}\cos\left\{(\rho+\tau)x - \frac{1}{2}(a+b+1)\pi\right\}$$

$$h_2 \sim \frac{1}{\pi\sqrt{\rho\tau x}}\cos\left\{(\rho-\tau)x - \frac{1}{2}(a-b)\pi\right\}$$

Difficulties

- $Y_n(x) \to -\infty$ as $x \to 0$, so split $\int_0^\infty$ into $\int_0^{ymax} + \int_{ymax}^\infty$ where $ymax = \max\{1\text{st zero of } Y_a(\rho x), 1\text{st zero of } Y_b(\tau x)\}$.

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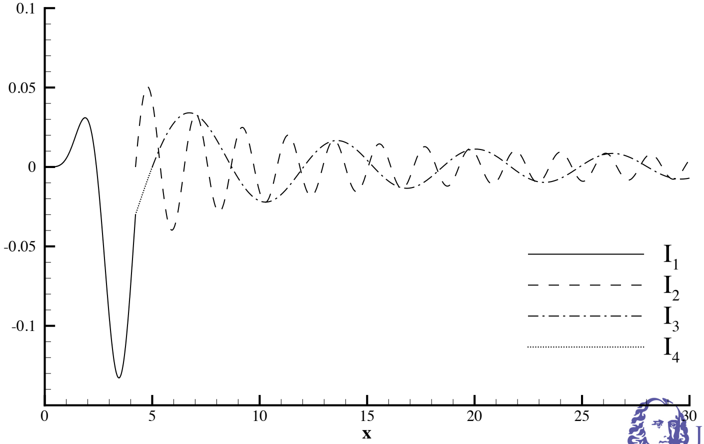
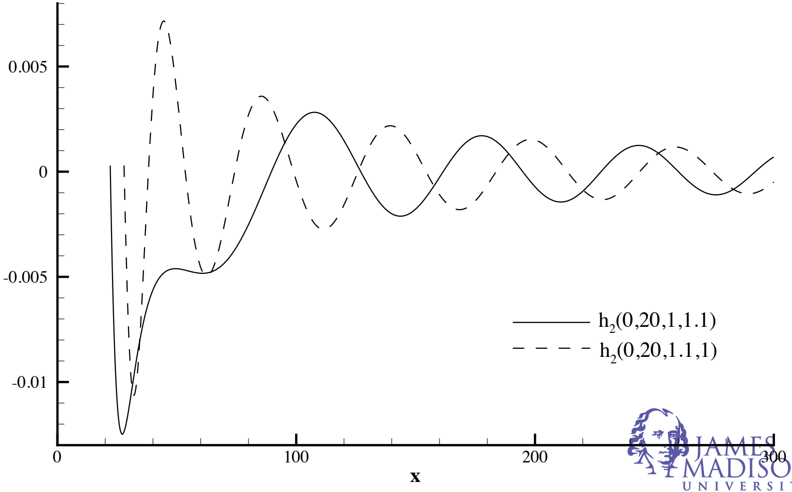
  - Use mW transform for h_1.

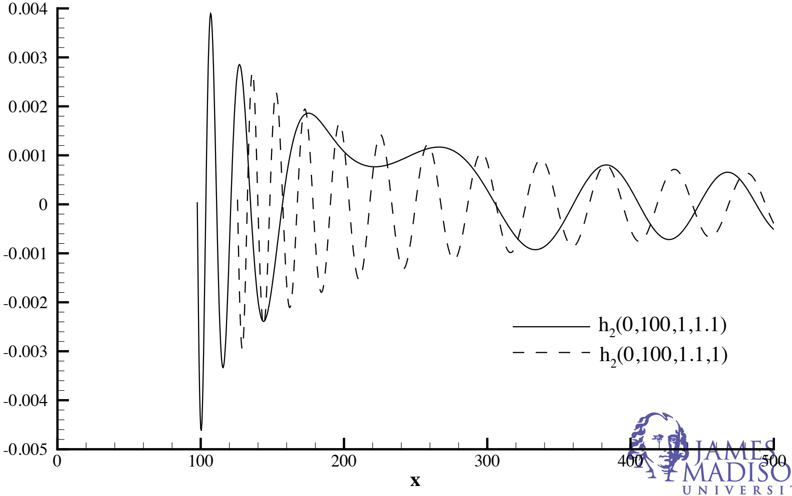
Transformation of $J_0(x)J_5(3x/2)$



h_2 When $\rho \sim \tau$ and a, b Are Far Apart



Even Worse


0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
-0.004
-0.005
0
100
200
300
400
500
x
$h_2(0,100,1,1.1)$
$h_2(0,100,1.1,1)$

Results

Excellent convergence rates. For example,

$$\int_0^\infty J_0(x)J_1(3x/2)\,dx = 2/3 \qquad \begin{array}{ll} \sim 200 \text{ evals, error} & \sim 10^{-5} \\ \sim 600 \text{ evals, error} & \sim 10^{-14} \end{array}$$

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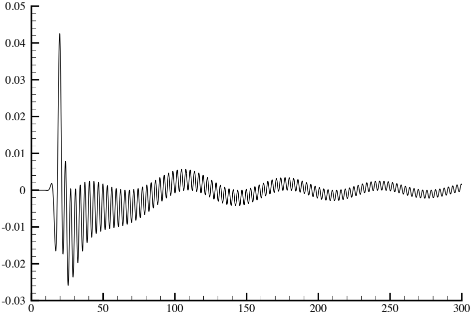

The Riemann Hypothesis

- The Riemann zeta function is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $(\mathcal{R}(s) > 1)$ or $\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$ $(\mathcal{R}(s) > 0)$.

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- The Riemann hypothesis states that all the non-trivial zeros of $\zeta(s)$ lie on the line $\mathcal{R}(s) = 1/2$.
- There are a variety of methods to more efficiently evaluate $\zeta(s)$, starting from the Euler-Maclaurin summation formula.

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Consider the x-ray plot of the Riemann zeta function – J. Arias-de-Reyna, X-ray of Riemann's zeta-function, *unpublished preprint*, 2003, http://arxiv.org/abs/math.NT/0309433

Clearly "difficult", not useful for analysis. But...

Riemann's ξ Function

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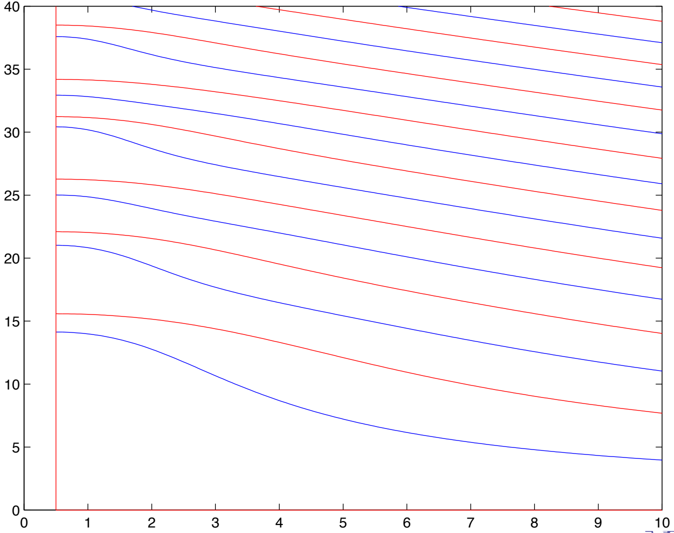
- After some manipulation, $(s = \sigma + it)$ we have

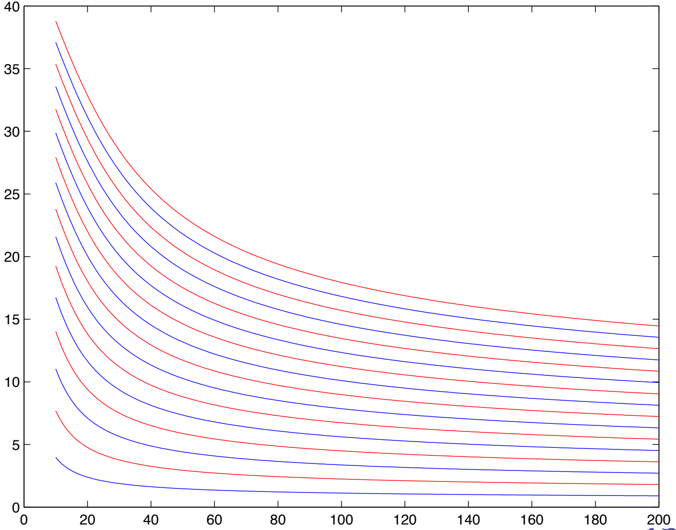
$$\begin{aligned}\xi(s) &= 8\pi \int_0^\infty \psi_2(y)\cosh((\sigma-1/2)y)\cos(ty)e^{5y/2}\,dy \\ &\quad + i8\pi \int_0^\infty \psi_2(y)\sinh((\sigma-1/2)y)\sin(ty)e^{5y/2}\,dy,\end{aligned}$$

where $\psi_2(y) = \sum_{n=1}^{\infty} a_n$ with $a_n = n^2\left(n^2 e^{2y}\pi - 3/2\right)e^{-n^2\pi e^{2y}}$.

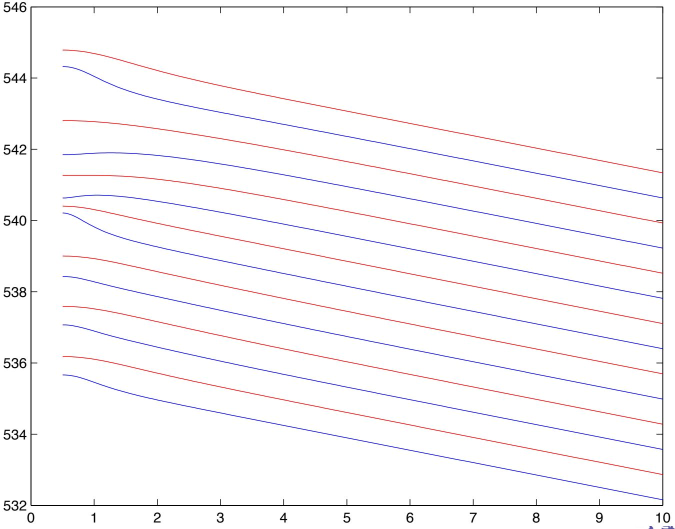
X-Ray for $\xi(s)$



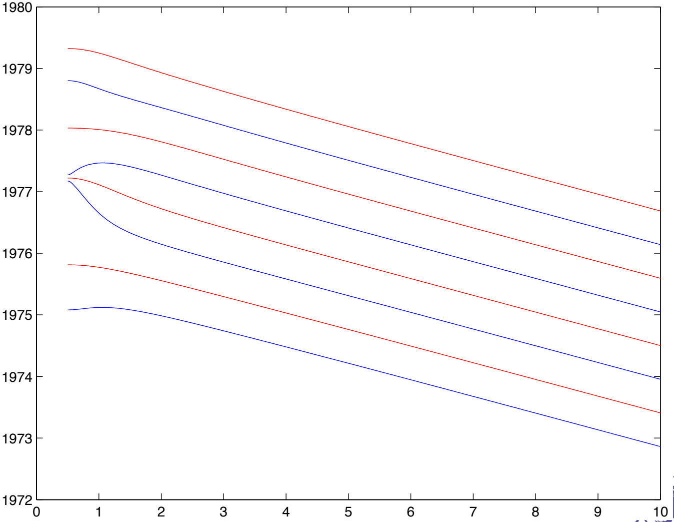
X-Ray for $\xi(s)$, Far Field



X-Ray for $\xi(s)$, Higher Up



X-Ray for $\xi(s)$, Even Higher Up



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Prove the real and imaginary integrals are not both zero simultaneously apart from when $\sigma = 1/2$.

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Prove the real and imaginary integrals are not both zero simultaneously apart from when $\sigma = 1/2$. Form implicit functions for blue and red curves. Can we bound the slopes for $\sigma = 1/2 + \epsilon$?

Filon Quadrature

- Real part is

$$8\pi \sum_{n=1}^{\infty} n^2 \int_0^{\infty} \left(n^2 e^{2y}\pi - \frac{3}{2}\right) e^{-n^2\pi e^{2y}} \cosh\left(\left(\sigma - \frac{1}{2}\right) y\right) \times \cos(ty) e^{5y/2}\, dy$$

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- Filon quadrature are increasingly accurate as t increases (Iserles 2003). It doesn't work.

Why Doesn't It Work?

$$8\pi \sum_{n=1}^{\infty} n^2 \int_0^{\infty} \left(n^2 e^{2y}\pi - 3/2\right) e^{-n^2\pi e^{2y}} \cosh((\sigma - 1/2)y) \cos(ty) e^{5y/2}\, dy$$

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- Why? $\xi(s)$ contains $\Gamma(s/2+1)$, which converges exponentially fast to zero as the imaginary part of s increases (in the critical strip).
- There are no asymptotic expansions for these integrals.
- Is there a symmetric function with the same zeros as $\zeta(s)$ which doesn't exponentially decay for large t? Perhaps generalizing the functional equation (Hill 2005)...

Help!

And thank you and any questions?