Computation for fun (not profit)

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Introduction

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- Math 448/449 cover classic topics of Numerical Analysis – solving equations, interpolation and approximation, numerical differentiation and integration, numerical linear algebra, numerical solution of differential equations etc.
- But computers programs can aid us in solving a much wider variety of problems.
- Today, we will look at a variety of problems (and their solutions) associated with continued fractions.

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The Euclidean Algorithm

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- Until $a_{n+1} = 0$. The greatest common divisor is then a_n.

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If gcd$(p, q) = 1$, we say p and q are *relatively prime*.

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Problem #1

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Repeat the calculation for $0 < p < q \leq 1000$.

Produce a graph of fraction of relatively prime pairs versus n, the upper bound on q. Does this suggest something to you about the fraction of numbers chosen at random that are relatively prime?

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Solution #1

Code for the gcd:

```
function ret=mygcd(p,q)
while q>0
  r=mod(p,q);
  p=q;
  q=r;
end
ret=p;
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Loop through fractions:

```
count=0;
for q=2:20
  for p=1:q-1
    if mygcd(p,q)==1
      count=count+1;
    end
  end
end
disp(count)
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Code returns 127 relatively prime pairs. Total number of pairs is $1 + 2 + 3 + \cdots + 19 = 190$, since $\sum_{i=1}^{n} i = n(n+1)/2$. The fraction is $127/190 \sim 0.668$.

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Solution #1 (cont'd)

Replacing 20 by 1000, $304\,191$ relatively prime pairs out of $499\,500$ in total. The fraction is $304191/499500 \sim 0.609$.

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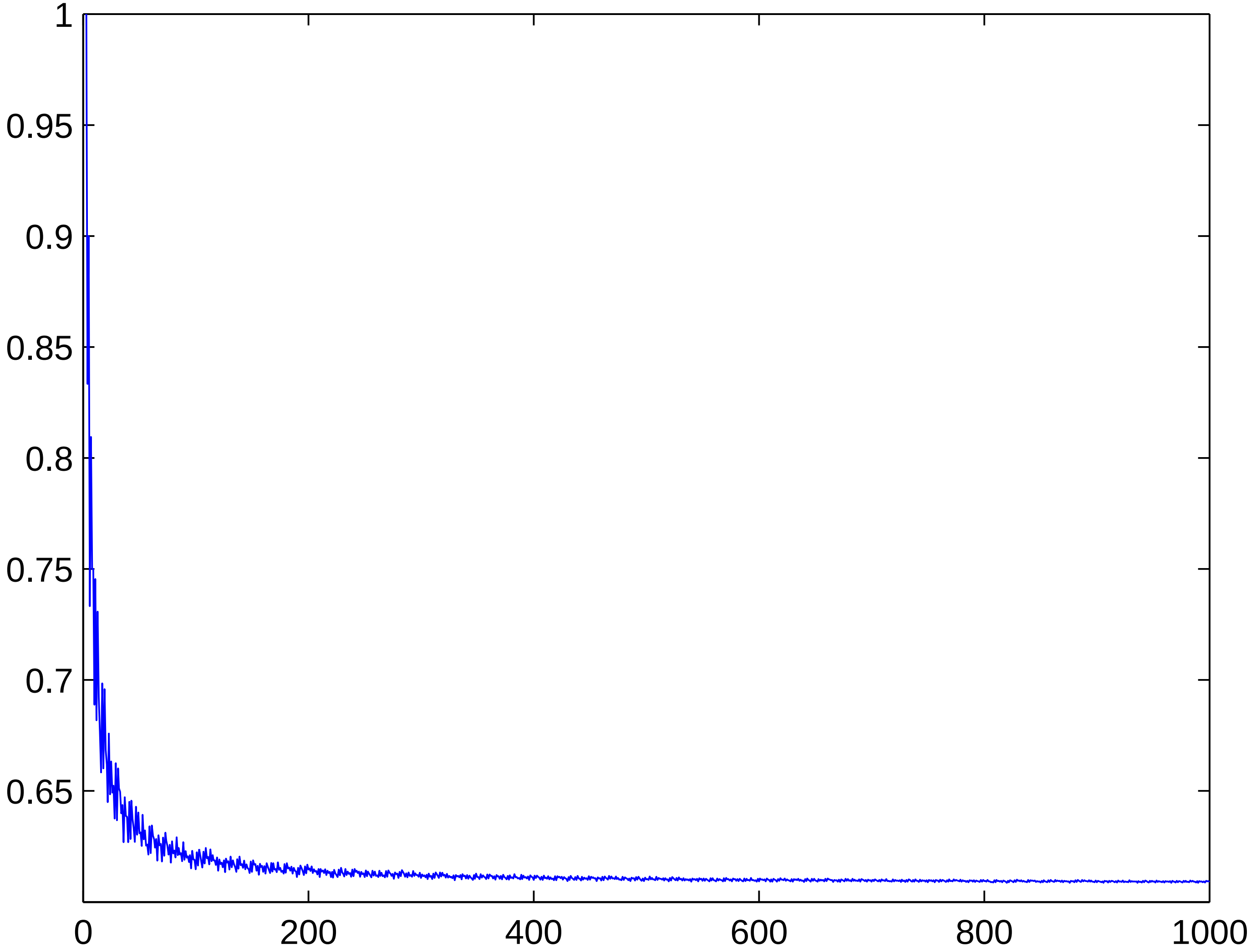
For lots of qs:

```
y=zeros(1,1000); count=0; total=0;
for q=2:1000
  for p=1:q-1
    if mygcd(p,q)==1, count=count+1; end
  end
  total=total+q-1; y(q)=count/total;
end
plot(2:1000,y(2:1000))
```

Solution #1 (cont'd)



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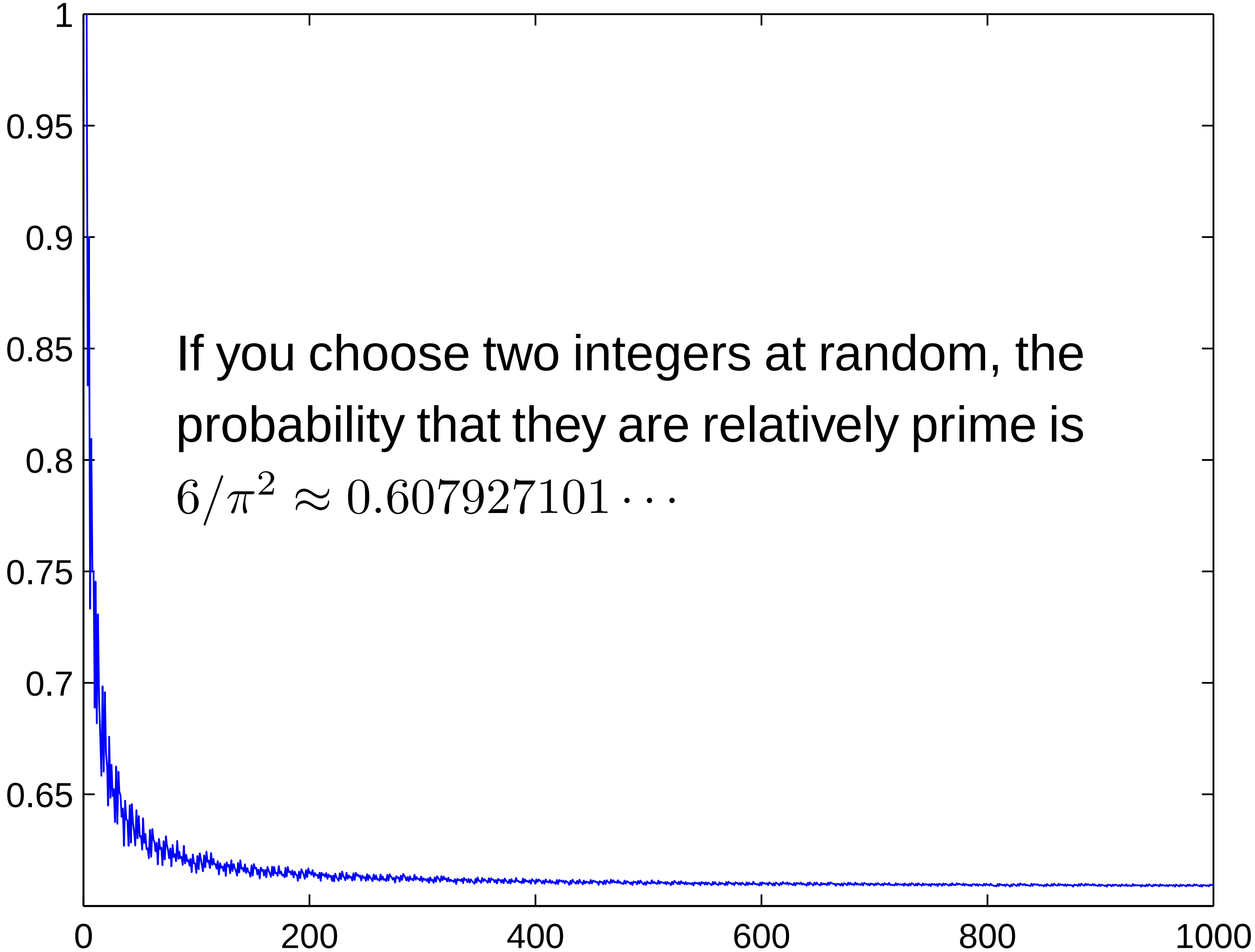

Continued Fractions

The iterative step of the Euclidian algorithm can be rewritten as

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and applied for $i = 0, 1, \ldots, n$ to give

$$\frac{p}{q} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{\vdots}{b_{n-1} + \cfrac{1}{b_n}}}}$$

where b_0 is an integer and the b_i's for $i > 0$ are positive integers. These are called *partial quotients*.

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$$\equiv b_0 + \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots + \frac{1}{b_n} \equiv [b_0; b_1, b_2, \ldots, b_n].$$

Continued Fraction Algorithm

- $x_0 = p/q$.
- $b_i = \lfloor x_i \rfloor$, $x_{i+1} = 1/(x_i - b_i)$ until $b_n = x_n$.

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```
function b=cf(p,q)
i=0;
while q~=0
  i=i+1; b(i)=floor(p/q);
  newp=q; newq=p-b(i)*q; fac=gcd(newp,newq);
  p=newp/fac; q=newq/fac;
end
```

Problem #2

For each fraction p/q, $0 < p < q \leq 500$ with $\text{gcd}(p, q) = 1$ (verify that there are $76\,115$ of them), we can find their continued fraction representations.

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- What is the maximum length of this set of continued fractions?
- What is the average length of this set of continued fractions?
- Which fractions have the form $[0; 1, 1, \cdots, 1, 2]$, where the number of ones could be zero one, two, etc.? Notice anything about them?

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Solution #2 – Code

```
lenb=0; tot=0; count=0;
for q=2:500
  for p=1:q-1
    if gcd(p,q)==1
      count=count+1; b=cf(p,q); lb=length(b);
      if lb>lenb
        lenb=lb; bigb=b; bigp=p; bigq=q;
      end
      tot=tot+lb;
      if max(b(2:lb-1)-1)==0 & b(lb)==2, disp([p,q]); end
    end
  end
end
disp([bigp,bigq,bigb]); disp([tot,count]), disp(tot/count);
```

Solution #2 – Results

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- Maximum length 13 for $233/377 = [0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]$.
- The average length is $478\,265/76\,115 \approx 6.283$.
- The fractions are $2/3$, $3/5$, $5/8$, $8/13$, $13/21$, $21/34$, $34/55$, $55/89$, $89/144$, $144/233$ and $233/377$.

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The gcd algorithm is slowest for successive Fibonacci numbers. Their continued fraction representations are the longest possible.

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The gcd algorithm is slowest for successive Fibonacci numbers. Their continued fraction representations are the longest possible.

Any continued fraction that ends in $[\cdots, 2]$ can also be written as ending in $[\cdots, 1, 1]$, so these longest continued fractions can in fact be represented as strings of ones!

Irrational Continued Fractions

The Euclidean algorithm leads to a finite continued fraction algorithm for rationals.

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The Euclidean algorithm leads to a finite continued fraction algorithm for rationals.

But the same algorithm can be used (indefinitely) for irrationals:

```
function b=realcf(x,n)
for i=1:n
  b(i)=floor(x); x=1/(x-b(i));
end
```

Problem #3

- Find the continued fraction approximations to $\ln(x)$ for $x = 2, 3, 4, \ldots, 10\,000$, with 11 partial quotients, i.e. $b_0, b_1 \ldots, b_{10}$.
- Ignoring the integer part b_0, we should have 99990 partial quotients.
- What fraction of these partial quotients are ones, two, three, etc. up to twenties, and what fraction are larger than twenty?

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Solution #3 – Code

```
count=zeros(21,1);
for i=2:10000
  b=realcf(log(i),11);
  for j=2:11
    if b(j)<=20, count(b(j))=count(b(j))+1;
    else, count(21)=count(21)+1;
    end
  end
end
count=count/sum(count);
fprintf('%10.6f\n',count);
```

Solution #3 – Results

n	Fraction		n	Fraction	
1	0.419602		11	0.010081	
2	0.163856		12	0.008991	
3	0.089239		13	0.007251	
4	0.057856		14	0.006181	
5	0.041944		15	0.005621	
6	0.030323		16	0.005101	
7	0.022982		17	0.004680	
8	0.018412		18	0.004470	
9	0.015062		19	0.003810	
10	0.012071		20	0.003220	
			> 20	0.069247	

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Solution #3 – Results

n	Fraction	Gauss-Kusmin	n	Fraction	Gauss-Kusmin
1	0.419602	0.415037	11	0.010081	0.010054
2	0.163856	0.169925	12	0.008991	0.008562
3	0.089239	0.093109	13	0.007251	0.007380
4	0.057856	0.058894	14	0.006181	0.006426
5	0.041944	0.040642	15	0.005621	0.005647
6	0.030323	0.029747	16	0.005101	0.005001
7	0.022982	0.022720	17	0.004680	0.004460
8	0.018412	0.017922	18	0.004470	0.004002
9	0.015062	0.014500	19	0.003810	0.003611
10	0.012071	0.011973	20	0.003220	0.003275
			> 20	0.069247	0.067114

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Solution #3 – Gauss-Kusmin

The Gauss-Kusmin (or Gauss-Kuzmin) theorem states that almost all irrationals have continued fractions whose partial quotients obey the rule that the number n occurs

$$-\frac{\ln\left(1-\frac{1}{(n+1)^2}\right)}{\ln(2)}$$

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$$-\frac{\ln\left(1-\frac{1}{(n+1)^2}\right)}{\ln(2)}$$

of the time.

This assumes we have the full infinite continued fraction. The agreement for this finite amount of data from the beginning of lots of different continued fraction is remarkable.

Continued Fraction Convergents

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They satisfy initially

$$p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = b_0, \quad q_0 = 1,$$

then

$$p_i = b_i p_{i-1} + p_{i-2} \quad \text{and} \quad q_i = b_i q_{i-1} + q_{i-2}.$$

for $i = 1, 2, \ldots$.

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for $i = 1, 2, \ldots$.

The numerator and denominator of convergents are always relatively prime, and have a variety of other interesting properties.

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- Find the fractions represented by $[1; 4, 8, 2, 12, 1]$ and $[2; 103, 1, 1, 2, 1]$.

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- The continued fraction for irrationals never finish, and can be approximated well by convergents. By generating convergents, can you estimate
  - $[1; 1, 2, 1, 2, 1, 2, \ldots]$,
  - $[2; 4, 4, 4, \ldots]$,
  - $[6; 1, 12, 1, 12, 1, 12 \ldots]$,
  - $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, \ldots]$, and
  - $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1 \ldots]$?

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  - $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, \ldots]$, and
  - $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1 \ldots]$?

  Note that the first three are periodic.

Solution #4 – Code

```
function x=evalcf(b)
p0=1; p1=b(1); q0=0; q1=1;
for i=2:length(b)
  p2=b(i)*p1+p0;
  q2=b(i)*q1+q0;
  p0=p1; p1=p2; q0=q1; q1=q2;
  disp([p1,q1,p1/q1]);
end
x=p1/q1;
```

Solution #4 – Results

- $[1; 4, 8, 2, 12, 1] = 1172/943 \simeq 1.24284199363733$
- $[2; 103, 1, 1, 2, 1] = 1457/725 \simeq 2.00965517241379$

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  - $[2; 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4] = 2.23606797749979$

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  - $[2; 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4] = 2.23606797749979 = \sqrt{5}$

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  - $[6; 1, 12, 1, 12, 1, 12, 1, 12, 1, 12, 1, 12, 1, 12, 1, 12] = 6.92820323027551$

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  - $[1; 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2] = 1.73205080756888 \; = \sqrt{3}$
  - $[2; 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4] = 2.23606797749979 = \sqrt{5}$
  - $[6; 1, 12, 1, 12, 1, 12, 1, 12, 1, 12, 1, 12, 1, 12, 1, 12] = 6.92820323027551 = \sqrt{48}$

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Solution #4 – Results

- $[1; 4, 8, 2, 12, 1] = 1172/943 \simeq 1.24284199363733$
- $[2; 103, 1, 1, 2, 1] = 1457/725 \simeq 2.00965517241379$
- For the next three, increase number of periodic pieces until solution doesn't change, or use the digits given. To 15 digits we get
  - $[1; 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2] = 1.73205080756888 \ = \sqrt{3}$
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  - $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1] = 2.71828182845905$

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Solution #4 – Results

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Problem #5 – $\sqrt{2}$

It appears square roots have periodic continued fractions. Can you prove $\sqrt{2} = [1; 2, 2, 2 \ldots]$?

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The convergents for $\sqrt{2}$ begin $1/1$, $3/2$, $7/5$, $17/12$, $41/29$, $99/70$, $239/169$, $577/408$, $1393/985$.

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In the first one thousand and one convergents, in how many of them does the numerator have more digits than the denominator?

Solution #5 – Periodic Solution

$\sqrt{2} = 1 + (\sqrt{2} - 1)$

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So $\sqrt{2} = [1; 2, 2, 2, \ldots]$.

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Solution #5 – Using Maple

The number of digits in x is $\lfloor \log_{10}(x) \rfloor + 1$.

Maple allows us to have arbitrarily large fractions (its just sometimes more difficult to program in than Matlab).

Code:

```
p(1):=3: q(1):=2: p(2):=7: q(2):=5: count:=0:
for i from 3 to 1000 do
  p(i):=2*p(i-1)+p(i-2); q(i):=2*q(i-1)+q(i-2);
  if floor(log10(p(i))) > floor(log10(q(i))) then
    count:=count+1:
  end if
end do;
count;
```

The solution is 153.

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Problem #6 – Periodic Cont. Fractions

All square roots of non-square numbers lead to periodic continued fractions.

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Using the overbar notation to indicate the periodic piece, the first few are $\sqrt{2} = [1; \overline{2}]$, $\sqrt{3} = [1; \overline{1\ 2}]$, $\sqrt{5} = [2\overline{4}]$, $\sqrt{6} = [2; \overline{2\ 4}]$, $\sqrt{7} = [2; \overline{1\ 1\ 1\ 4}]$, and so on.

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It turns out that exactly four continued fractions for $\sqrt{n}$ have odd period for $n \leq 13$ and non-square. How many continued fractions for $\sqrt{n}$ have odd period for $n \leq 10\,000$ and non-square?

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Solution #6 – Fraction Algorithm

The continued fraction algorithm is $x_0 = \sqrt{n}$, then $a_i = \lfloor x_i \rfloor$, $x_{i+1} = 1/(x_i - b_i)$, stop when you recognize periodicity.

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If $x_i = a_i + (b_i\sqrt{n} + c_i)/d_i$, then $x_{i+1} = \dfrac{d_i}{b_i\sqrt{n} + c_i}$

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If $x_i = a_i + (b_i\sqrt{n} + c_i)/d_i$, then $x_{i+1} = \dfrac{d_i}{b_i\sqrt{n} + c_i} = \dfrac{b_i d_i \sqrt{n} - c_i d_i}{b_i^2 n - c_i^2}$,

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Solution #6 – Code

```
function [a,left,right]=sqrtcf(n)
if mod(sqrt(n),1)==0
  a=sqrt(n); left=0; right=-1;
else
  b(1)=0; c(1)=1; d(1)=1; sn=sqrt(n); done=0; i=0;
  while ~done
    i=i+1; a(i)=floor((b(i)+c(i)*sn)/d(i));
    b(i+1)=(b(i)-a(i)*d(i))*d(i); c(i+1)=-c(i)*d(i);
    d(i+1)=(b(i)-a(i)*d(i))^2-c(i)^2*n;
    t=gcd(gcd(b(i+1),c(i+1)),d(i+1));
    if t>1, b(i+1)=b(i+1)/t; c(i+1)=c(i+1)/t; d(i+1)=d(i+1)/t; end
    found=0; j=i+1;
    while ~found && j>1
      j=j-1; found=(b(i+1)==b(j))&(c(i+1)==c(j))&(d(i+1)==d(j));
    end
    if found, left=j; right=i; done=1; end
  end
end
```

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    found=0; j=i+1;
    while ~found && j>1
      j=j-1; found=(b(i+1)==b(j))&(c(i+1)==c(j))&(d(i+1)==d(j));
    end
    if found, left=j; right=i; done=1; end
  end
end
```

The solution is 1322.

Problem #7 – e

The number e has the surprisingly regular continued fraction
$[2; 1, 2, 1,\ 1, 4, 1,\ 1, 6, 1,\ \ldots\ \ 1, 2k, 1,\ \ldots]$,
surprising because almost all irrational numbers have partial quotients with no discernable patterns.

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$[2; 1, 2, 1,\ 1, 4, 1,\ 1, 6, 1,\ \ldots\ \ 1, 2k, 1,\ \ldots]$,
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The first few convergents of e are $2/1$, $3/1$, $8/3$, $11/4$, $19/7$, $87/32$, $106/39$, $193/71$, $1264/465$ and $1457/536$. The sum of the digits in the numerator of the tenth convergent is $1 + 4 + 5 + 7 = 17$.

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What is the sum of the digits of the numerator of the one hundredth convergent of e?

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Solution #7

High precision arithmetic is again required. Let's resort to Maple, concentrating on the numerator only.

```
p(1):=2: p(2):=3: p(3):=8: p(4):=11:
for i from 2 to 33 do
  j:=3*i-1; p(j):=p(j-1)+p(j-2);
  j:=j+1; p(j):=2*i*p(j-1)+p(j-2);
  j:=j+1; p(j):=p(j-1)+p(j-2);
end do:

x:=p(100); s:=0:
while x>0 do
  s:=s+irem(x,10); x:=iquo(x,10);
end do:
s;
```

The answer is 272.

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There are no solutions when D is a square. The first few minimal solutions in x for $D \leq 7$ are $3^2 - 2 \times 2^2 = 1$, $2^2 - 3 \times 1 = 1$, $9^2 - 5 \times 4^2 = 1$, $5^2 - 6 \times 2^2 = 1$ and $8^2 - 7 \times 2^2 = 1$. The largest x for this set is 9, when $D = 5$.

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Find the value of $D \leq 1000$ whose minimal solution in x has the largest value of x.

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Find the value of $D \leq 1000$ whose minimal solution in x has the largest value of x.

A brute force search is impractical. Can we relate this to continued fractions?

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The solution is $D = 661$ with
$x = 164\,21658\,24296\,59102\,75055\,84047\,22704\,71049$!

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Subtract the integer part, leaving the fractional part.

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Subtract the integer part, leaving the fractional part.

What is the first fraction p/q (in lowest form) that isn't in this set, where we order fractions in ascending order in q then p?

Problem #9

So let's look at a simpler problem – representing rationals.

Generate all the bounded continued fractions in $F(4) + F(4)$ with $b_0 = 0$ and up to n additional partial quotients, and form the set of every possible sum of two numbers from these sets.

Subtract the integer part, leaving the fractional part.

What is the first fraction p/q (in lowest form) that isn't in this set, where we order fractions in ascending order in q then p?

How does the fraction change an n increases?

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Solution #9

Its actually much faster to consider the fractions (in lowest form) in order $(1/2, 1/3, 2/3, 1/4, 3/4, 1/5, \ldots)$, and in each case split into two fractions (in order) and check if they are both in $F(4)$! Identify the best for each fraction in terms of number of terms.

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n	Fraction
1	$\frac{1}{5} = [0; 4, 1]$
2	$\frac{1}{7} = -\frac{4}{7} + \frac{5}{7} = [-1; 23] + [0; 122]$
3	$\frac{2}{13} = -\frac{8}{13} + \frac{10}{13} = [-1; 2112] + [0; 133]$
4	$\frac{4}{23} = -\frac{14}{23} + \frac{18}{23} = [-1; 2114] + [0; 13112]$
5	$\frac{13}{37} = -\frac{16}{37} + \frac{29}{37} = [0; 11341] + [0; 131112]$
6	$\frac{22}{73} = -\frac{80}{219} + \frac{2}{3} = [-1; 1112144] + [0; 12]$
7	$\frac{51}{121} = -\frac{113}{143} + \frac{333}{1573} = [-1; 41332] + [0; 4121111124]$

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Conjecture:

Any rational can be represented by the sum of two **finite** elements of $F(4)$.

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Unfortunately, the proof is elusive...

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Conclusion

- Continued fractions have some lovely properties – consider a course in number theory.

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Conclusion

- Continued fractions have some lovely properties – consider a course in number theory.
- Computation can be used to solve (or give insight into) problems in many areas of mathematics.
- As an alternative to classic Numerical Analysis, consider Project Euler...