A Direct Proof of the Prime Number Theorem

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The primes . . . those exasperating, unruly integers that refuse to be divided evenly by any integers except themselves and one.

Martin Gardner

Outline

- Why me?
- What is the Prime Number Theorem.
- History of Prime Number Theorem.
- Transforms.
- A direct proof.
- Riemann’s and the “exact” forms.
- Number of distinct primes.
- Conclusion.

Thanks to Ken Lever (Cardiff), Richard Martin (London)

Research Projects

Numerical Analysis

Accurately finding multiple roots.

Euler-Maclaurin-like summation for Simpson's rule.

Placing a circle pack.

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New

Improving kissing sphere bounds.

Counting cycles in graphs.

Integer-valued logistic equation.

Fitting data to predator-prey.

Representing reals using bounded continued fractions.

Analysis of “Dreidel”.

The Prime Number Theorem

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Another form states that

$$\pi(x) \sim \int_0^x \frac{dt}{\ln t} \qquad (= \mathrm{li}(x))$$

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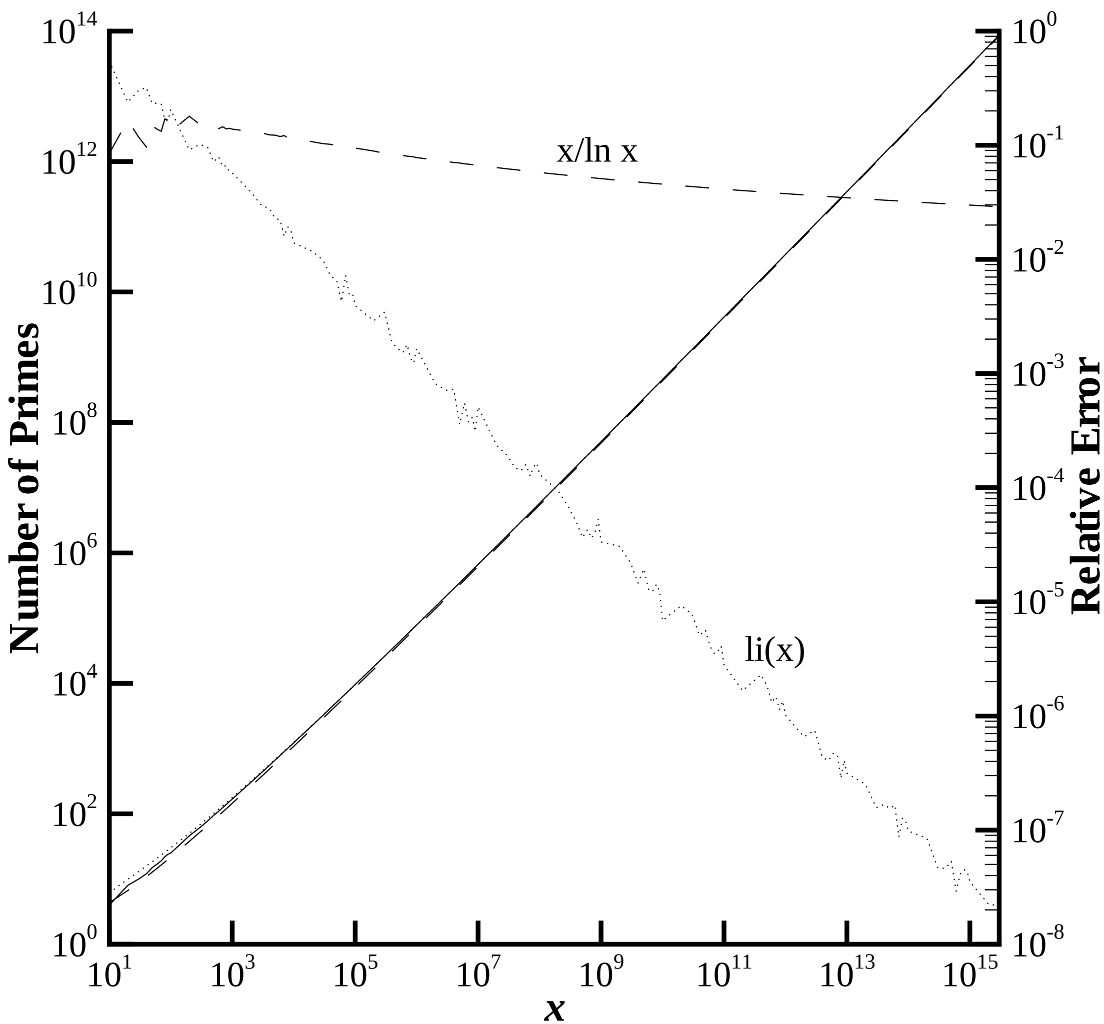
Plot of $\pi(x)$, $x/\ln x$, li(x)



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- Gauss (1849, as early as 1792) used numerical evidence to conjecture that
$$\pi(x) \sim \int_0^x \frac{dt}{\ln t}.$$
- Dirichlet (1837) introduced Dirichlet series:
$$\widehat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

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History of the PNT (cont'd)

- Chebyshev (1851) introduced

$$\theta(x) = \sum_{p \le x} \ln p, \qquad \psi(x) = \sum_{p^m \le x} \ln p,$$

and showed that the PNT is equivalent to

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 1, \qquad \lim_{x \to \infty} \frac{\psi(x)}{x} = 1.$$

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- Riemann (1860) introduced the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad \Re(s) > 1.$$

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$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \ln\left(1 - x^{-2}\right),$$

where ρ are the non-trivial zeros of $\zeta(s)$, so the PNT is equivalent to

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- Riemann hypothesis: $\Re(\rho) = \frac{1}{2}$.

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First Proof of the PNT

- Hadamard & de la Valèe Poussin (1896, independently) showed $\exists a, t_0$ such that $\zeta(\sigma + it) \neq 0$ if $\sigma \geq 1 - \dfrac{1}{a \log |t|}$, $|t| \geq t_0$, so

$$\psi(x) = x + O\left(xe^{-c(\log x)^{1/14}}\right).$$

Improvements to the PNT Proof

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- Ikehara (1930) & Wiener (1932) used Tauberian theorems to prove Ikehara's theorem:
  - Let $\widehat{f}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $\{a_i\}$ real and nonnegative.
  - If $\widehat{f}(s)$ converges for $\Re(s) > 1$, and $\exists A > 0$ s.t. for all $t \in \mathbb{R}$, $\hat{f}(s) - \frac{A}{(s-1)} \to$ finite limit as $s \to 1^+ + it$, then $\sum_{n \leq x} a_n \sim Ax$.
  - This can be used to show $\lim_{x \to \infty} \psi(x)/x = 1$ directly.

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- Newman (1980) showed $\lim_{x \to \infty} \psi(x)/x = 1$ using straightforward contour integration.
- Lucas, Martin & Lever – current work, looks at $\pi(x)$ directly.

Laplace Transforms

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- $\mathcal{L}^{-1}\{\bar{f}(s)\} = H(x)f(x) = \dfrac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \bar{f}(s) e^{sx}\, ds.$
  - H is the unit step function.
  - ϵ to the right of any singularities in $\bar{f}(s)$.
  - assume $\bar{f}(s) \to 0$ as $|s| \to \infty$.

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  - H is the unit step function.
  - ϵ to the right of any singularities in $\bar{f}(s)$.
  - assume $\bar{f}(s) \to 0$ as $|s| \to \infty$.
- Choose $\bar{g}(s)$ such that $\bar{f}(s) - \bar{g}(s)$ is analytic at the rightmost singularity in $\bar{f}(s)$. If $\bar{g}(s) \to 0$ as $|s| \to \infty$, then shift the integration contour to the left, apply the Cauchy integral theorem, and get that $f(x) = g(x) + O(x^c)$, where c is the new position of ϵ. (e.g. Smith, 1966)

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Arithmetic Functions and Dirichlet Series

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Given that

$$\frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{x^s}{s}\, ds = \begin{cases} 0, & x < 1, \\ 1, & x > 1, \end{cases} \qquad \epsilon > 0,$$

then

$$\begin{aligned} \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} x^s \widehat{f}(s) \frac{ds}{s} &= \sum_{n=1}^{\infty} f(n) \times \begin{cases} 0, & x < n, \\ 1, & x > n, \end{cases} \\ &= \sum_{1 \le n \le x} f(n). \end{aligned}$$

The Transform

We can recognize this as the inversion of a Laplace-like transform $(x = e^t)$:

$$\begin{aligned} (\mathcal{M}f)(t) &= \int_1^\infty f(x) x^{-s-1}\, dx, \\ H(x-1)f(x) &= \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} (\mathcal{M}f)(t) x^t \, dt. \end{aligned}$$

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Some useful transform pairs are:

f $\quad\overset{\mathcal{M}}{\longleftrightarrow}$	$\mathcal{M}(f)$
$\gamma + \ln(\ln x)$	$\frac{1}{s}\log\frac{1}{s}$
$x^a \mathsf{li}(x^c)$	$\frac{1}{s-a}\log\frac{c}{s-a-c}, \quad \Re(s) > c > 0$
$\Gamma(k)^{-1} x^c (\ln x)^{k-1}$	$\frac{1}{(s-c)^k}, \quad \Re(s) > c;\ k > 0$

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Average Order Asymptotics

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and $x\widetilde{f}(x) \overset{\mathcal{M}}{\longleftrightarrow} \dfrac{\widehat{f}(s)}{s}$ and $\widetilde{f}(x) \overset{\mathcal{M}}{\longleftrightarrow} \dfrac{\widehat{f}(s+1)}{s+1}$.

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So, to find the asymptotic form of an average order:

- find its Dirichlet series $\widehat{f}(s)$ in closed form,
- identify the position and form of the singularities in $\dfrac{\widehat{f}(s)}{s}$,
- sum the inverse transforms of the singular parts.

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Riemann and Prime Zeta-Functions

The Riemann zeta-function is the Dirichlet series of $f(n) = 1$:

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The prime zeta-function is the Dirichlet series of

$$f(n) = \begin{cases} 1, & n \text{ prime}, \\ 0, & \text{otherwise}, \end{cases} \quad : \quad P(s) = \sum_p \frac{1}{p^s}.$$

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The Prime Zeta-Function

Now

$$\log\zeta(s)=\sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_p\sum_{m=1}^{\infty}\frac{1}{mp^{ms}}$$

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Using the Möbius inversion formula,

$$P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns),$$

where $\mu(n) = \begin{cases} 0, & n \text{ contains a square factor,} \\ (-1)^m, & n \text{ a product of } m \text{ distinct primes.} \end{cases}$

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So $\pi(x) \sim$ li(x).

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So

$$\pi(x) \sim \text{li}(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \text{li}\left(x^{1/n}\right) = R(x).$$

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Zero of $\zeta(s)$ at $s = \rho_m$ means a singularity in $P(s)/s$ at $s = \rho_m$ of the form $\frac{1}{s} \log\left(\frac{s}{\rho_m} - 1\right)$, whose inverse is $-\mathrm{li}(x^{\rho_m})$.

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As for Riemann's form, singularity in $\zeta(s)$ at $s = \rho_m$ also means singularities in $P(s)$ at $s = \rho_m/n$, $\mu(n) \neq 0$. These singularities are of the form $\frac{\mu(n)}{ns} \log\left(\frac{ns}{\rho_m} - 1\right)$, whose inverses are $-\mu(n)\mathrm{li}\left(x^{\rho^m/n}\right)/\,n$.

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The contributions of all singularities related to $s = \rho_m$ contribute $-R(x^{\rho_m})$ to $\pi(x)$, and so

$$\pi(x) = \lim_{k\to\infty} R_k(x) \quad \text{where} \quad R_k(x) = R(x) - \sum_{m=-k}^{k} R(x^{\rho_m}).$$

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If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ (p_i's some primes), then $\omega(n) = k$ and $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

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It is well known (e.g. Hardy & Wright) that $\widetilde{\omega}(n) = \ln(\ln n) + B_1 + o(1)$, where $B_1 = \gamma + \sum_p \left\{ \ln\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} = 0.26149\,72128\,47642\dots$,

and $\widetilde{\Omega}(n) = \widetilde{\omega}(n) + \sum_p \frac{1}{p(p-1)} = \widetilde{\omega}(n) + B_2 - B_1$,

where $B_2 = 1.03465\,38818\,97438\dots$.

Asymptotics for ω

We can show that

$$\widehat{\omega}(s) = \zeta(s)P(s), \quad \text{and} \quad \widetilde{\omega}(x) \stackrel{\mathcal{M}}{\longleftrightarrow} \frac{\zeta(s+1)}{(s+1)}P(s+1).$$

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$$\widehat{\omega}(s) = \zeta(s)P(s), \quad \text{and} \quad \widetilde{\omega}(x) \xleftrightarrow{\mathcal{M}} \frac{\zeta(s+1)}{(s+1)}P(s+1).$$

Expanding around the rightmost singularity at the origin,

$$\frac{1}{s}\log\frac{1}{s} + \frac{1}{s}\sum_{m=2}^{\infty}\frac{\mu(m)}{m}\ln\zeta(m) + \frac{\gamma-1}{s+1}\log\frac{1}{s} + O(1).$$

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Previously $\widetilde{\omega}(n) \sim \ln(\ln n) + \gamma + \sum_{p}\left\{\ln\left(1-\frac{1}{p}\right) + \frac{1}{p}\right\}$.

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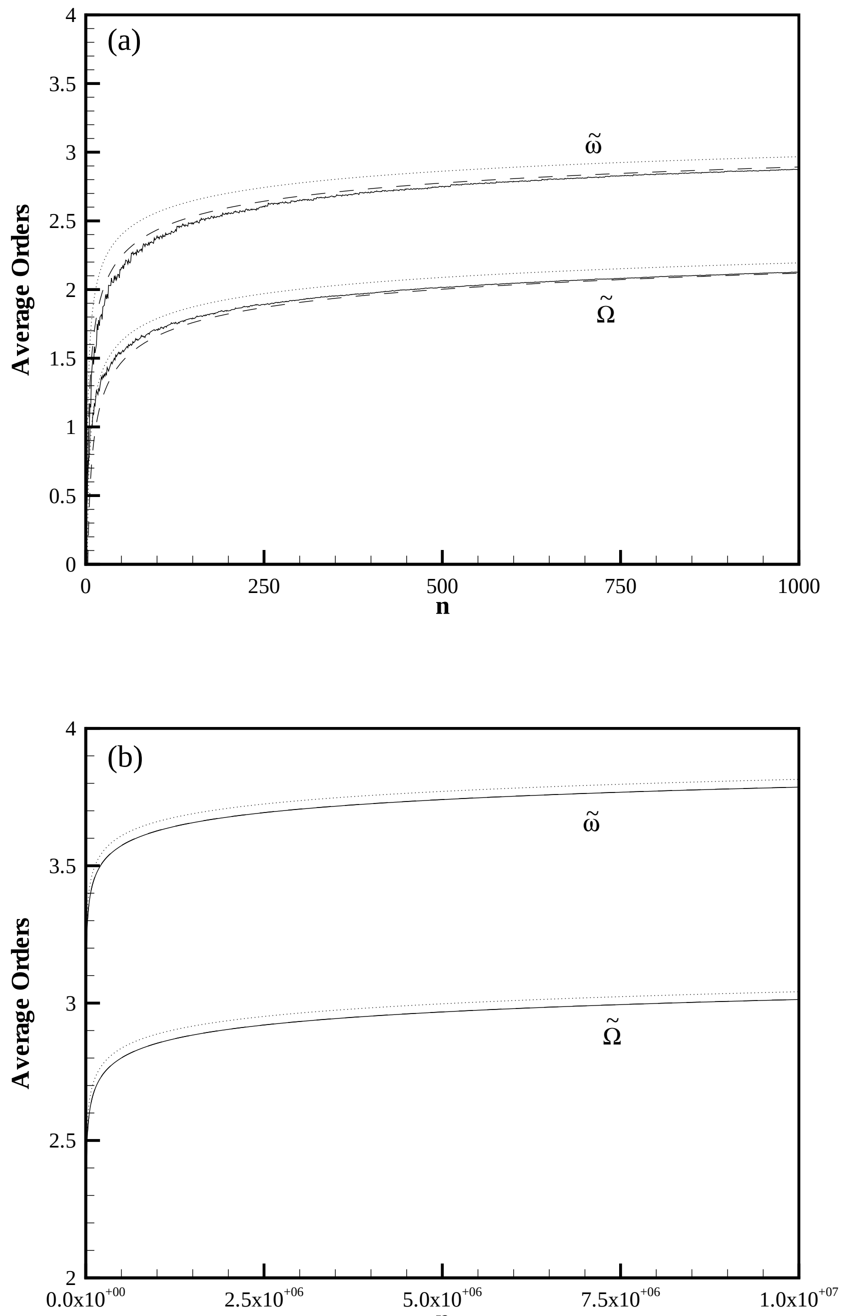
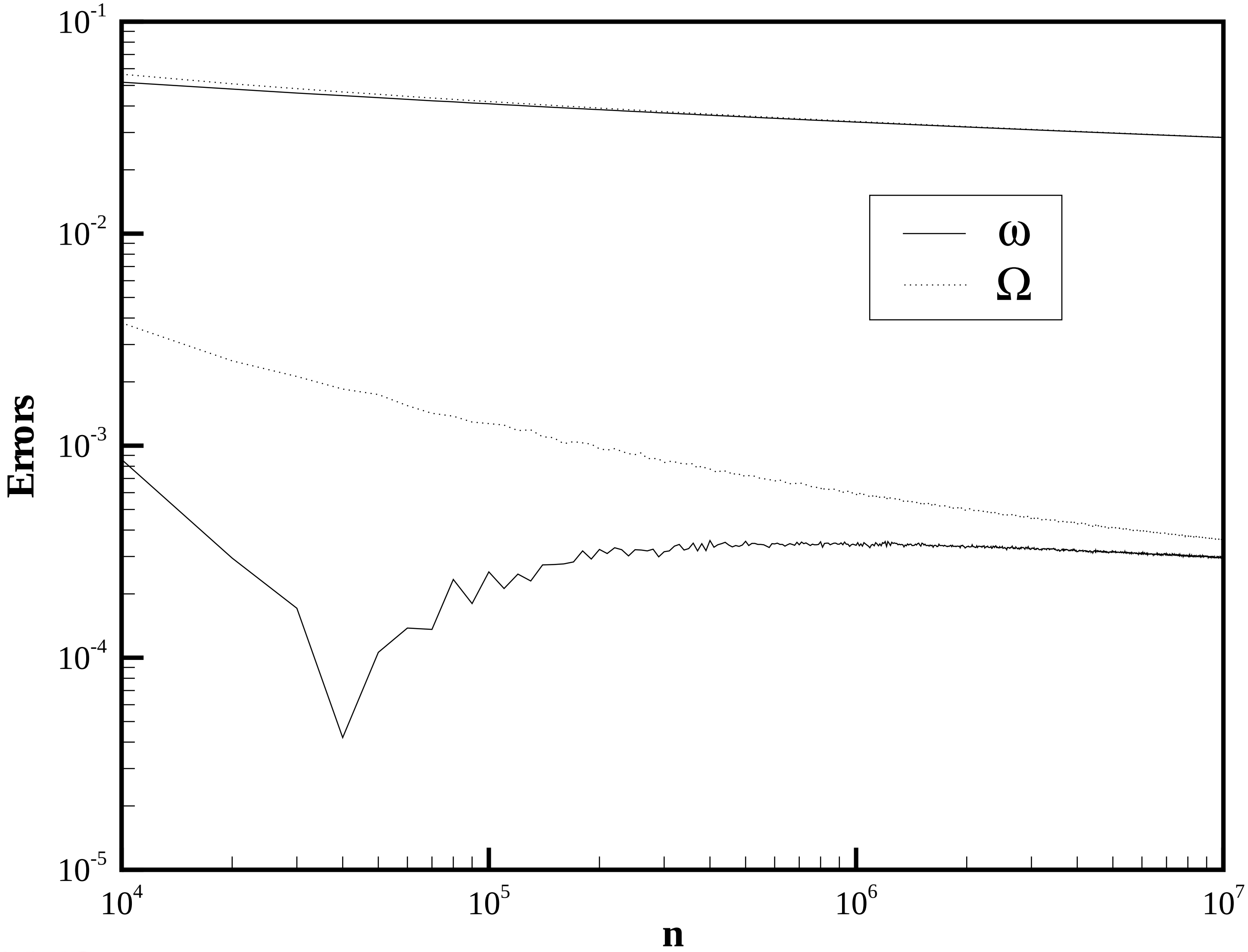
$$\left(\text{Old } \widetilde{\Omega}(n) = \widetilde{\omega}(n) + \sum_{p} \frac{1}{p(p-1)} \right).$$

Results for ω and Ω



Asymptotic Errors for ω and Ω



Conclusion

We have developed a generalization of Ikehara's theorem that relates the asymptotic behavior of the average order of an arithmetic function to the singularities in its Dirichlet series.

We have used this technique to prove the prime number theorem directly, without recourse to $\psi(x)$, derived both the Riemann and "exact" forms of $\pi(x)$, and found a correction to the classical ω, Ω average order asymptotics.

Further work:

- Apply technique to other arithmetic functions whose Dirichlet functions are known in closed form. Will the results improve the classical results?
- Reformulation of the twin prime conjecture:
  - $\pi_2(x)$ is the number of twin primes between 1 and x, and is x times the average order of $t(n)$, where $t(n) = p(n)p(n-2)$
  - Can we use results for p to find the Dirichlet series for t, find the form of the rightmost singularity of $\widehat{t}$, and find a result relating this to the conjectured asymptotic $\pi_2(x) \sim \dfrac{2Cx}{(\log x)^2}$, where $C = \prod_{p \geq 3} \dfrac{p(p-2)}{(p-1)^2}$?

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An Integral Proof That $\pi < 355/113$

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Unfortunately, no integers m, n lead to $I_{m,n}$ involving other continued fractions for π.

However,

$$\int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{3164(1+x^2)}\,dx = \frac{355}{113} - \pi,$$

which proves $\pi < 355/113$.

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