

# MATH 236 (FALL 2014) QUIZ III ON CHAPTER 8

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Attempt all problems. Box your answers.

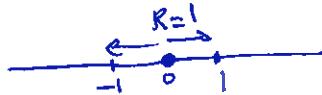
(1) Consider the series

$$1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

(a) Find the radius of convergence, interval of convergence, and the type of convergence on that interval.

$$\lim_{n \rightarrow \infty} \sqrt[n]{|(-1)^n x^n|} = \lim_{n \rightarrow \infty} |x| = |x| \quad \text{series conv. absolutely when } |x| < 1$$

$$\Rightarrow -1 < x < 1$$



→ Radius of conv. = 1.

endpts: for  $x=1$   $\sum_{n=0}^{\infty} (-1)^n$  does not conv.  
 since  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n \neq 0$

$$\text{for } x=-1 \quad \sum_{n=0}^{\infty} (-1)^n (-1)^n = \sum_{n=0}^{\infty} 1 = \infty \text{ dev.}$$

concl: series conv. abs. on the interval  $(-1, 1)$

- (b) The above series is also a geometric series. What is the common ratio  $r$ ? What is the exact sum of the above series? (Hint: Use the formula  $\frac{r^{\text{first index}}}{1-r}$ ).

$$r = -x \quad \text{series adds up to } \frac{r^0}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x} \quad \text{for } |r| = |-x| = |x| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots = \frac{1}{1+x} \quad \text{for } |x| < 1$$

(Conversely, starting with the  $f(x) = \frac{1}{1+x}$  & computing its Taylor expansion about  $x_0 = 0$  gives  $\sum_{n=0}^{\infty} (-1)^n x^n$ )

- (c) Deduce from part (b) the identity

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

and find the interval on which it is valid (check the endpoints individually).

The above argument shows that  $\frac{1}{1+x}$  is analytic on the interval  $(-1, 1)$

$$X \quad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots \quad \text{on } (-1, 1)$$

We can integrate the above power series, within its interval of convergence:  
term by term

let  $s \in (-1, 1)$ ,

$$\int_0^s \frac{1}{1+x} dx = \sum_{n=0}^{\infty} \int_0^s (-1)^n x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{s^{n+1}}{n+1} = s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + \dots$$

$$\Rightarrow \ln(1+s) - \ln(1+0) = s - \frac{s^2}{2} + \frac{s^3}{3} - \dots$$

$$\Rightarrow \ln(1+s) = s - \frac{s^2}{2} + \frac{s^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^n s^n}{n}$$

The interval of conv. of the new series is the same as the one we started with,  
except possibly at the endpoints.

So series conv. absolutely on  $(-1, 1)$ , & let's check endpoints:

at  $s=1$ :  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges (alternating harmonic series: alt. test:  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  • an +ve  
• an ↓ •  $\lim_{n \rightarrow \infty} a_n = 0$ )  
convergence here is only conditional

$$\text{since } \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

at  $s=-1$ :  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} = -\infty$

Therefore  $\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \text{ on the interval } (-1, 1)}$

- (d) How many terms from the above series do you need to use to be able estimate  $\ln(2)$  to within 0.001 of its value? What is that value?

for  $x=1$  above, we have

$$\ln(1+1) = \ln(2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad \text{where } a_n = \frac{2^n}{n}$$

this is an alternating series, we have the error estimate if we truncate after  $n$ -terms

$$\left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} - (a_1 - a_2 + a_3 - a_4 + \dots + (-1)^n a_n) \right| < a_{n+1} < 0.001$$

$$\Rightarrow \frac{2^{n+1}}{n+1} < 0.001$$

$$\Rightarrow n+1 > \frac{1}{0.001} = 1000$$

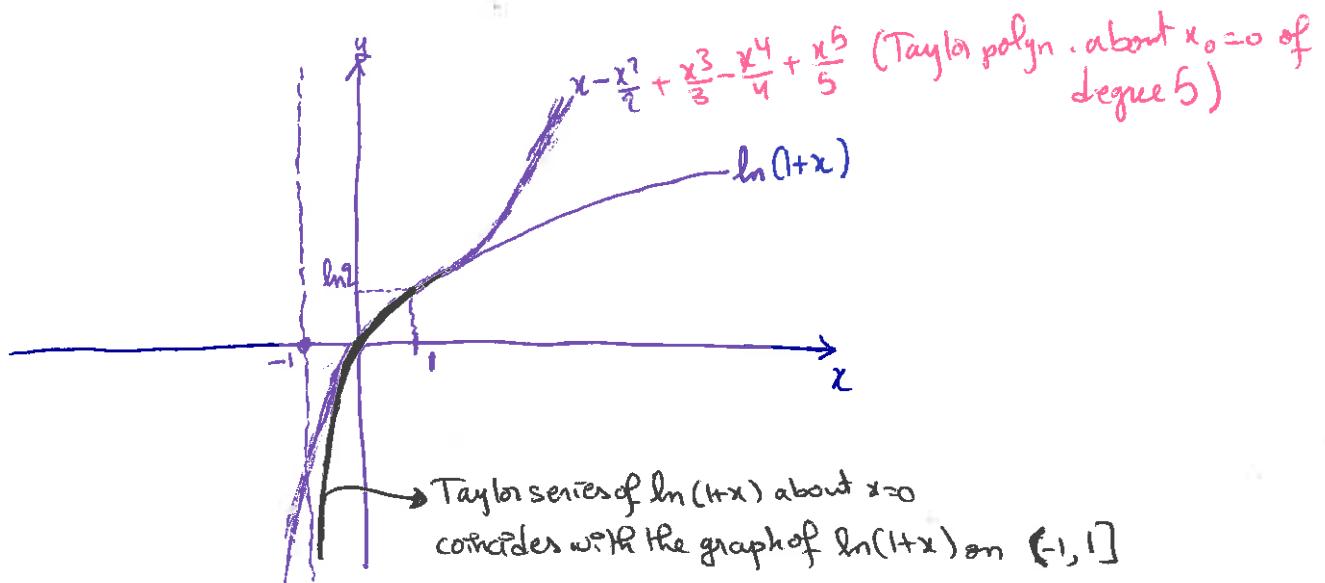
$$\Rightarrow n > 999$$

$\Rightarrow$  We need to use 1000 terms from the series to be able to estimate  $\ln(2)$  to within 3 digits  
 ↓ accurate (so the convergence of the series to  $\ln 2$  is very slow)

The exact value of  $\ln 2 = 0.69314718\dots$

so we need to use 1000 terms of the series to get only the first 3 digits!!!  
 ↓ at least

- (e) Plot a (very neat) graph of  $\ln(1+x)$ , its Taylor polynomial of order 5, and of its series expansion (all on the same graph). Specify clearly the interval on which the series is well-defined.



object 1: the fn  $\ln(1+x)$  on  $[-1, \infty)$  ← its domain

object 2: Taylor polyn. of degree 5:  $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$  ← on  $(-\infty, +\infty)$

object 3: Taylor series expansion of  $\ln(1+x)$  about  $x=0$ :  $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$  ← on  $(-1, 1]$   
 coincides with  $\ln(1+x)$  on this interval

(2) Write down the definition of:

(a) (i)  $f$  is smooth on the interval  $(a, b)$ .

$f(x)$  is smooth on  $(a, b)$  iff it has infinitely many cont. derivatives at all pts of the interval  $(a, b)$ , that is, its derivatives of all orders exist & are cont. ■

(ii)  $f$  is analytic on the interval  $(a, b)$ .

$f(x)$  is analytic on  $(a, b)$  iff its Taylor expansion  $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$  around each pt  $x_0$  in  $(a, b)$  converges to  $f(x)$  in some nbhd containing  $x_0$ .

(b) Is every real valued smooth function analytic?

No, see the fn  $\{e^{-1/x^2}\}$  on the next page.

Every smooth fn has a Taylor expansion but that Taylor expansion need not converge to the fn  $f(x)$ .

- (c) Show that the function  $e^{-1/x^2}$  and its Taylor series expansion around  $x_0 = 0$  disagree everywhere except at  $x = 0$ . Plot a figure to illustrate. Is this function analytic anywhere?

Taylor expansion of  $e^{-1/x^2}$  about  $x_0=0$  is  $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

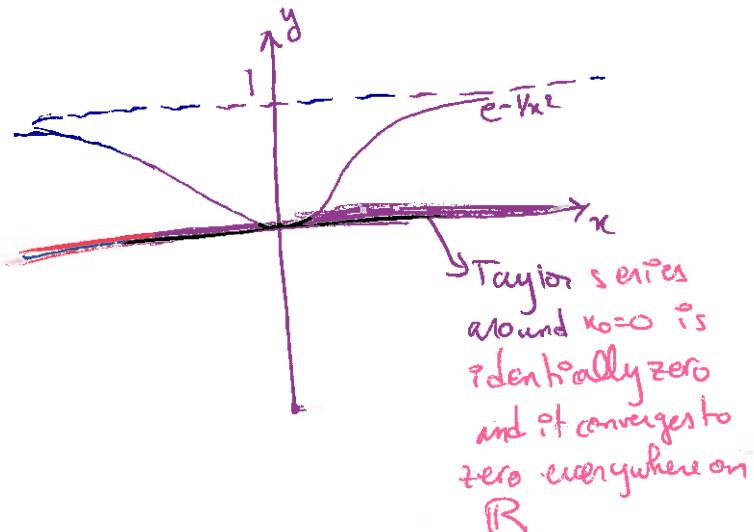
$$f(0) = \lim_{x \rightarrow 0} e^{-1/x^2} = 0$$

$$f'(0) = \lim_{x \rightarrow 0} \frac{d}{dx} e^{-1/x^2} = 0$$

$$f''(0) = \lim_{x \rightarrow 0} \left( \frac{4-6x^2}{x^6} \right) e^{-1/x^2} = 0$$

⋮

(The exponential term kills all the other polyn. terms as  $x \rightarrow 0$ )



Hence, Taylor expansion of  $e^{-1/x^2}$  around  $x_0=0$  is identically ZERO! (since all the derivatives are zero at  $x_0=0$ )

but  $e^{-1/x^2}$  is only zero at  $x=0$ , and is positive ( $>0$ ) otherwise.

Hence  $e^{-1/x^2}$  & its Taylor expansion around  $x_0=0$  disagree everywhere except at  $x=0$ .

$e^{-1/x^2}$  is not analytic on any interval containing zero, but analytic everywhere else.

- (3) (a) Prove that the function  $\frac{\sin x}{x}$  is analytic on  $\mathbb{R}$  and find its Taylor expansion around  $x_0 = 0$ .

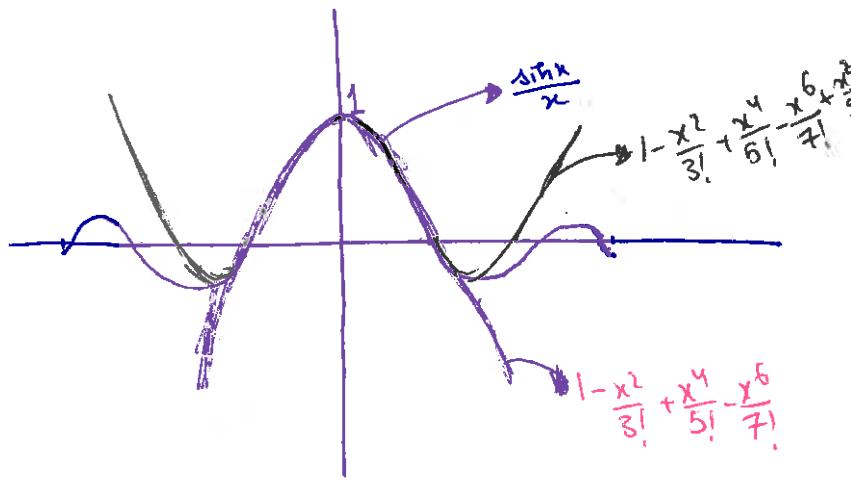
Start with the analytic fn  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$  on  $\mathbb{R}$

Divide by  $x \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$  on  $\mathbb{R}$

$$\text{Hence } \frac{\sin x}{x} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-1)!} \text{ on } \mathbb{R}$$

(so  $f_n$  is the Taylor series expansion around  $x_0 = 0$  for all  $n \in \mathbb{N}$   
 $\Rightarrow$  thus  $f_n$  is analytic on  $\mathbb{R}$ )

- (b) Plot  $\frac{\sin x}{x}$  and its Taylor polynomial approximations of order 6 and 8, on the interval  $(-10, 10)$ .



object 1:  $f_n \frac{\sin x}{x}$

object 2: Taylor poly. of order 6:

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$$

object 3: Taylor poly. of order 8:

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$

- (c) What must you change if we were to consider the function  $\frac{\cos x}{x}$  instead?

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\text{so } \frac{\cos x}{x} = \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \dots$$

has trouble at  $x=0$

Hence, we should be away from  $x=0$   
 $\cos x$  has a singularity at  $x=0$  called a pole  
but it is not a singularity for  $\frac{\cos x}{x}$   
We can expand away from  $x_0 = 0$ .  $\frac{\cos x}{x}$  is not analytic on all of  $\mathbb{R}$

(4) Find all values of  $x$  for which the following series converges

$$\sum_{n=0}^{\infty} \frac{x^n}{1+n2^n}.$$

When is the convergence absolute and when is it conditional?

$$\text{Compute } \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1}{1+n2^n} \right| |x|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{1+n2^n}} |x|$$

$$= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n2^n}} |x|$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n2^n}} |x|$$

$$= \frac{1}{2} |x|$$

absolute convergence when

$$\frac{1}{2} |x| < 1 \Rightarrow |x| < 2$$

$\Rightarrow$  radius of conv. = 2.

$$\begin{array}{c} \leftarrow \quad \rightarrow \\ -2 \quad 0 \quad +2 \end{array}$$

Check endpts:

$$\underline{x=2}: \sum_{n=0}^{\infty} \frac{2^n}{1+n2^n} \sim \sum_{n=0}^{\infty} \frac{1}{n} \text{ div. harmonic series (p=1)}$$

$$\text{Justify "N" by limit comparison: } \lim_{n \rightarrow \infty} \frac{\frac{2^n}{1+n2^n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n2^n}{1+n2^n} = 1$$

$$\underline{x=-2}: \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{1+n2^n}$$

This is an alternating sequence  $\sum_{n=0}^{\infty} (-1)^n a_n$

where  $a_n$ 's are positive

decreasing

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2^n}{1+n2^n} = \lim_{n \rightarrow \infty} \frac{2^n}{n2^n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

} Hence by alt.  
series test,  
it converges  
(conditionally).

Therefore, the series  $\sum_{n=0}^{\infty} \frac{x^n}{1+n2^n}$  converges absolutely on  $(-2, 2)$  and conditionally at  $x=\pm 2$

Answer: convergence on  $[-2, 2]$  and absolute conv. on  $(-2, 2)$

Review for final, do not solve Using the Lagrange form of the Taylor remainder  $R^n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ , prove that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \text{for all } x \in \mathbb{R}$$

Start with  $f(x) = \sin x$ .

Its Taylor expansion about  $x_0=0$  is  $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$f(x) = \sin x \quad f(0) = 0 \quad \text{get} \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f''''(x) = \sin x \quad f''''(0) = 0$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$$

$\vdots$

Now its Taylor remainder is

$$\frac{f^{(n+1)}(c)(x)^{n+1}}{(n+1)!} = \left\{ \sin c, \cos c, -\sin c, -\cos c \right\} x^{n+1}$$

$\rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in \mathbb{R}$

since the numerator is ~~not~~ bounded

(and  $x^{n+1}$  is dominated by  $(n+1)!$  as  $n \rightarrow \infty$ )

Therefore, since  $\sin x = P_{x_0=0}^n(x) + \text{Taylor remainder } R^n(x)$

take  $\lim_{n \rightarrow \infty}$  we get  $\sin x = \text{Taylor series} + \underbrace{\lim_{n \rightarrow \infty} (\text{Taylor remainder } R^n(x))}_{\rightarrow 0 \text{ as } n \rightarrow \infty}$

$\Rightarrow \sin x = \text{its Taylor series about } x_0=0 \text{ for all } x \in \mathbb{R}$

$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \text{ for all } x \in \mathbb{R}$