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MATH 236 (FALL 2014) QUIZ II ON CHAPTER 7

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Attempt all problems. Box your answers.

- (1) Study the convergence of the following series. Specify the type of convergence for series with positive and negative terms (conditionally or absolutely convergent). Justify all your answers.

2 (a) $\sum_{n=3}^{\infty} \frac{\sqrt{n+2} - \sqrt{n-2}}{n}$

$$= \sum_{n=1}^{\infty} \frac{(\sqrt{n+2} - \sqrt{n-2})(\sqrt{n+2} + \sqrt{n-2})}{n(\sqrt{n+2} + \sqrt{n-2})}$$

$$= \sum_{n=1}^{\infty} \frac{n+2 - (n-2)}{n(\sqrt{n+2} + \sqrt{n-2})} = \sum_{n=1}^{\infty} \frac{4}{n(\sqrt{n+2} + \sqrt{n-2})} \sim \sum_{n=1}^{\infty} \frac{4}{n^{3/2}}$$

converges

p-series with
 $p = \frac{3}{2} > 1$

Justify " \sim ": use limit comparison

$$\lim_{n \rightarrow \infty} \frac{\frac{4}{n(\sqrt{n+2} + \sqrt{n-2})}}{\frac{4}{n^{3/2}}} = \lim_{n \rightarrow \infty} \frac{4n^{3/2}}{4n(\sqrt{n+2} + \sqrt{n-2})} = 1$$

2 (b) $\sum_{n=1}^{\infty} (\sin \frac{1}{n})^{1+\epsilon}$ where $0 < \epsilon < 1$.

for large n , $\frac{1}{n}$ is small, so $\sin \frac{1}{n} \approx \frac{1}{n}$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\sin \frac{1}{n} \right)^{1+\epsilon} \sim \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^{1+\epsilon} = \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$$

conv. p-series
 $p = 1 + \epsilon > 1$

Justify " \sim " use limit comparison:

$$\lim_{n \rightarrow \infty} \frac{\left(\sin \frac{1}{n} \right)^{1+\epsilon}}{\left(\frac{1}{n} \right)^{1+\epsilon}} = \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right)^{1+\epsilon} = \lim_{m \rightarrow 0} \left(\frac{\sin m}{m} \right)^{1+\epsilon} = 1^{1+\epsilon} = 1$$

let $m = \frac{1}{n}$

$\left(\lim_{m \rightarrow 0} \frac{\sin m}{m} = 1 \right)$

(note that $\sin \frac{1}{n}$ is positive for large n)

$\sum_{n=3}^{\infty} \frac{2}{n\sqrt[3]{n}\ln n}$

$\sqrt[3]{n} \rightarrow 1$ as $n \rightarrow \infty$ so $\sum_{n=3}^{\infty} \frac{2}{n\sqrt[3]{n}\ln n} \sim \sum_{n=3}^{\infty} \frac{2}{n\ln n}$ diverges by integral test

Now justify " \sim " by a limit comparison

$$\lim_{n \rightarrow \infty} \frac{\frac{2}{n\sqrt[3]{n}\ln n}}{\frac{2}{n\ln n}} = \lim_{n \rightarrow \infty} \frac{2n\ln n}{2n\sqrt[3]{n}\ln n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[3]{n}} = 1$$

$$\int_3^{\infty} \frac{2}{x\ln x} dx = 2 \int_3^{\infty} \frac{du}{u \ln u}$$

$$= 2 \ln |\ln u| \Big|_3^{\infty}$$

$a(x) = \frac{2}{x\ln x}$ is \rightarrow positive
 \rightarrow decreasing
 \rightarrow cont.

so integral test applies

$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^{3/2}}$

Since $|\cos^2(n)| \leq 1$

then $\frac{|\cos^2(n)|}{n^{3/2}} \leq \frac{1}{n^{3/2}}$ & $\sum \frac{1}{n^{3/2}}$ conv. (p -series, $p = 3/2 > 1$)

then by "actual" comparison, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ converges as well

2 (e) $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} - \dots$

$$= 1 - \left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) - \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right) + \left(\frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} \right) - \dots$$

$$= 1 - \frac{5}{6} + \frac{37}{60} - \frac{1207}{2520} + \frac{7793}{20020} - \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $\{a_n\}$ is a sequence which is

- positive
- decreasing
- ↓ & $\lim_{n \rightarrow \infty} a_n = 0$

⇒ by alternating series test, the series converges

(only conditional convergence since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges: harmonic series)

- (2) **Fractal: Koch snowflake** In Figure 1, the area inside the Koch snowflake can be described as the union of infinitely many equilateral triangles. Each side of a smaller triangle is exactly one third the size of a side of the large one. If the area if the inner most triangle is 1 unit of area, find the total area of the Koch snowflake.

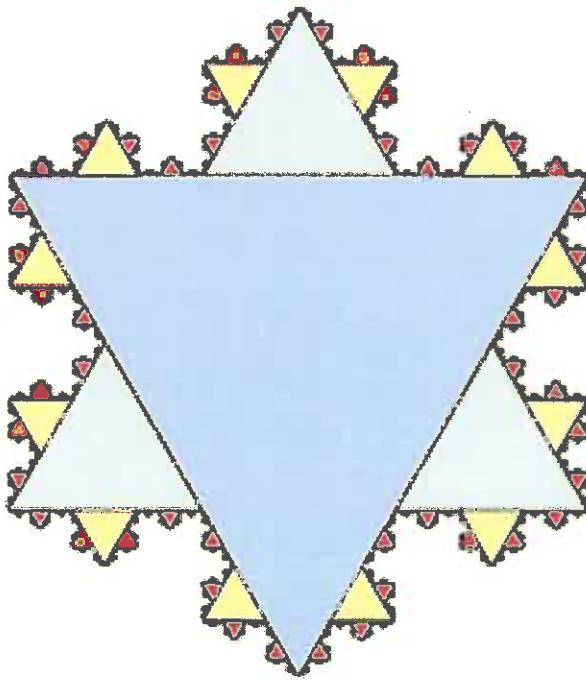


FIGURE 1. Koch snowflake (source: Wikipidea)

$$1^{\text{st}} \text{ triangle (innermost)}: \text{area} = \frac{\sqrt{3}}{4} a^2$$

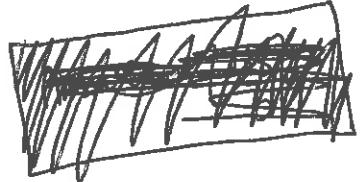
$$2^{\text{nd}} \text{ layer}: \text{area} = 3 \times \frac{\sqrt{3}}{4} \left(\frac{1}{3}a\right)^2 = \frac{\sqrt{3}}{4} a^2 \cdot 3 \times \frac{1}{9}$$

3 triangles
in that
layer

side now is
 $\frac{1}{3}$ previous side

$$3^{\text{rd}} \text{ layer}: \text{area} = 12 \times \frac{\sqrt{3}}{4} \left(\frac{1}{3}\frac{1}{3}a\right)^2 = \frac{\sqrt{3}}{4} a^2 \cdot 12 \left(\frac{1}{9}\right)^2$$

$$4^{\text{th}} \text{ layer}: \text{area} = 48 \times \frac{\sqrt{3}}{4} \left(\frac{1}{3}\frac{1}{3}\frac{1}{3}a\right)^2 = \frac{\sqrt{3}}{4} a^2 \cdot 48 \left(\frac{1}{9}\right)^3$$



$$\text{Total area} = \frac{\sqrt{3}}{4} a^2 + \frac{\sqrt{3}}{4} a^2 \cdot 3 \times \frac{1}{9} + \frac{\sqrt{3}}{4} a^2 \cdot 12 \left(\frac{1}{9}\right)^2 + \frac{\sqrt{3}}{4} a^2 \cdot 48 \left(\frac{1}{9}\right)^3 + \dots$$

$$= \frac{\sqrt{3}}{4} a^2 \left[1 + 3 \times \frac{1}{9} + 12 \left(\frac{1}{9}\right)^2 + 48 \left(\frac{1}{9}\right)^3 + \dots \right]$$

$$= \frac{\sqrt{3}}{4} a^2 \left[1 + 3 \times \frac{1}{9} + 3 \times 4 \left(\frac{1}{9}\right)^2 + 3 \times 4^2 \left(\frac{1}{9}\right)^3 + \dots \right]$$

$$= \frac{\sqrt{3}}{4} a^2 \left[1 + \frac{3}{9} \left\{ 1 + \frac{4}{9} + \frac{4^2}{9^2} + \dots \right\} \right]$$

$$= \frac{\sqrt{3}}{4} a^2 \left[1 + \frac{3}{9} \cdot \frac{9}{5} \right] = \frac{\sqrt{3}}{4} a^2 \left[1 + \frac{3}{5} \right] = \left(\frac{\sqrt{3}}{4} a^2 \right) \frac{8}{5}$$

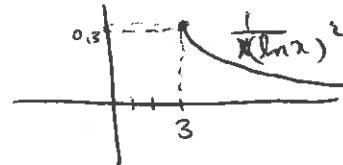
but the inner most triangle has
area = 1
⇒ Area = $\frac{8}{5}$ unit of
area

$\frac{1}{1 - \frac{4}{9}} = \frac{1}{\frac{5}{9}} = \frac{9}{5}$ (geometric series)

- (3) Prove that the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent, and estimate the answer up to 6 accurate digits (you want to use enough terms in the finite sum so that the error due to truncation is less than 10^{-6}). Hint: Use the error bound from the integral test.

The fn $\frac{1}{x(\ln x)^2}$ is

- cont. on $[3, \infty)$
- decreasing on $[3, \infty)$
- positive on $[3, \infty)$



\Rightarrow the integral test applies &

$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ and $\int_3^{\infty} \frac{dx}{x(\ln x)^2}$ both converge or both diverge.

$$\int_3^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 3}^{\infty} \frac{du}{u^2} = -\frac{1}{u} \Big|_{\ln 3}^{\infty} = 0 + \frac{1}{\ln 3} = \frac{1}{\ln 3} < \infty$$

$\Rightarrow \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the integral test.

To get 10^{-6} truncation error, we have the error estimate

$$\left| \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} - (a_3 + a_4 + \dots + a_n) \right| < \int_n^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln n} < 10^{-6}$$

$$\Rightarrow \ln n > 10^6$$

$$\Rightarrow n > e^{10^6} \text{ (unrealistic)}$$

so this series converges extremely slowly!

- (4) **Fixed point iteration:** We will find a root of the function $f(x) = x^4 - x - 10$ using a fixed point iteration instead of Newton's method (saves us the trouble of differentiating $f(x)$).

2.5

We want to solve $f(x) = 0$ which we can write as $x = (x + 10)^{\frac{1}{4}}$ or $x = g(x)$ where $g(x) = (x + 10)^{\frac{1}{4}}$. Hence, a root of $f(x)$ is a *fixed point* of $g(x)$. Just like we did in Newton's method, we will construct a sequence of numbers that hopefully will converge to a fixed point of $g(x)$, which will be a root of $f(x)$.

- Start with initial guess $x_0 = 1$.
- Construct a sequence using the recurrence relation $x_{k+1} = g(x_k)$, for $k = 0, 1, 2, \dots$
- Stop when $|x_{k+1} - x_k| < \text{tolerance}$, where tolerance = 10^{-5} .
- Check that your last iterate x_m is indeed a fixed point of $g(x)$ (so $x_m \approx g(x_m)$) and hence a root of $f(x)$ (so $f(x_m) \approx 0$).

$$x_0 = 1$$

$$x_1 = g(x_0) = (1+10)^{\frac{1}{4}} = 1.82116$$

$$x_2 = g(x_1) = (1.82116+10)^{\frac{1}{4}} = 1.85424$$

$$x_3 = g(x_2) = (1.85424+10)^{\frac{1}{4}} = 1.85553$$

$$x_4 = g(x_3) = (1.85553+10)^{\frac{1}{4}} = 1.85558$$

$$x_5 = g(x_4) = (1.85558+10)^{\frac{1}{4}} = 1.85558$$

stop: $|x_5 - x_4| = 0 < 10^{-5}$

check: $x_5 = g(x_5) \rightarrow 1.85558 \approx (1.85558+10)^{\frac{1}{4}}$

$$f(x_5) = x_5^4 - x_5 - 10 = (1.85558)^4 - (1.85558) - 10 = -0.00011207 \approx 10^{-4}$$

- (5) (a) Consider the sequence $\{\sin \frac{n\pi}{6}\}$.

(i) Is it bounded? What are the least upper bound and the greatest lower bound?

0.5

yes, bd by $-1 \leq \sin \frac{n\pi}{6} \leq 1$. l.u.b = 1, g.l.b = -1, they are achieved.

$$\left\{ \sin \frac{\pi}{6}, \sin \frac{2\pi}{6}, \sin \frac{3\pi}{6}, \sin \frac{4\pi}{6}, \sin \frac{5\pi}{6}, \sin \frac{6\pi}{6}, \sin \frac{7\pi}{6}, \sin \frac{8\pi}{6}, \sin \frac{9\pi}{6}, \sin \frac{10\pi}{6}, \dots \right\}$$

$$\left\{ \sin \frac{\pi}{6}, \sin \frac{\pi}{3}, \sin \frac{\pi}{2}, \sin \frac{2\pi}{3}, \sin \frac{5\pi}{6}, \sin \pi, \sin \left(\pi + \frac{\pi}{6}\right), \sin \left(\pi + \frac{2\pi}{6}\right), \sin \left(\pi + \frac{3\pi}{6}\right), \sin \left(\pi + \frac{4\pi}{6}\right) \right\}$$

$$\left\{ \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, \dots \right\}$$

0.5

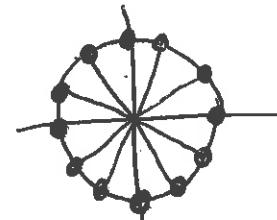
(ii) Is it convergent? Does it have any convergent subsequences? How many subsequential limits can you identify? What are they?

- Not convergent (more than one subsequential limit)

- It has convergent subsequences.

There are 7 subsequential limits

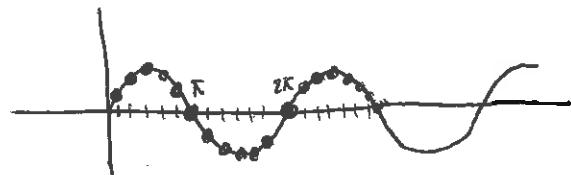
$$1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$$



0.5

(iii) Is it monotonic? Why or why not?

Not monotonic (sin oscillates)



0.5

(iv) Does it have a divergent subsequence?

No, it's bd (if it had a divergent subsequence it would be unbd.)

0.5

(b) Is the corresponding series $\sum_{n=1}^{\infty} \sin \frac{n\pi}{6}$ convergent? Why or why not?

No, since the sequence of the terms $\sin \frac{n\pi}{6} \rightarrow 0$ as $n \rightarrow \infty$