Math 236 (Fall 2014) Final Exam

Wed Dec 10, 2014

Name:

Honor Pledge: I understand that it is a violation of the JMU honor code to give or receive unauthorized aid on this exam. Furthermore, I understand that I am obligated to report any violation of the honor code by other students that I may become aware of, and that my failure to do so is itself a violation. No phones, or any electronic devices other than a graphing calculator may be accessed during this test. Doing so will be considered a violation of the honor code.

You are allowed to use a graphing calculator, and to have only one (letter size) page (one side only) of your own notes.

Signature: ______________________________

Look for the Table of Famous Analytic Functions on the last page of this exam. Attempt all problems. Box your answers.

(1) Water is evaporating from an open vase at a rate

$$r(t) = -0.05\pi(10 - 0.2t)^2$$

cubic centimeters per hour. The vase initially contains 200 cm^3 of water. How much time will it take for all the water in the vase to evaporate?

(2) Why is the integral $\int_0^1 \sin\left(\frac{1}{x^2}\right)\, dx$ improper (where is the trouble)? Prove that it converges. (Hint: Let $u = \frac{1}{x^2}$, then use comparison to a convergent integral.)

(3) Find the volume of a solid obtained by revolving the region enclosed by the graph of $y = \sqrt{\cos(x)}$, $y = 0$ and the lines $x = \frac{\pi}{4}$ and $\frac{\pi}{2}$ around the x-axis.

(4) A solid has the circle with radius 2, $x^2 + y^2 = 4$, as its base. Its cross-sections, taken perpendicular to the base and to the x-axis are squares.

(a) Find its volume. (Hint: Each slice is a square whose area is $(side)^2$, and has an infinitesimal thickness dx.)

(b) Draw a three-dimensional scheme showing the shape of this solid (welcome to Math 237!).

(5) Use the integral test to prove that the series $\sum_{n=0}^{\infty} e^{-n}$ converges, then find the least number of terms we need to use in order to estimate the sum to within 10^{-6} of its value. If your n is reasonable, find that estimate.

(6) (a) Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

is convergent.

(b) Use partial fractions to show that the above series is telescoping, and find its exact sum.

(7) Consider the sequence $\{(-1)^n 2^{1/n}\} = \{-2, 2^{1/2}, -2^{1/3}, 2^{1/4}, \ldots\}$

(a) How many sub-sequential limits does this sequence have? What are they?

(b) Is this sequence convergent? Why or why not.

(c) Is it bounded? If yes, identify the least upper bound and the greatest lower bound.

(d) Is the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ convergent? Justify.

(e) How about the series $\sum_{n=1}^{\infty} (-1)^n (1 - 2^{1/n})$? Justify.

(8) Consider the improper integral

$$\int_0^\infty \frac{e^{-xt} - e^{-t}}{t} dt,$$

where $x > 0$.

(a) Why is this integral improper? Split the integral into the sum of two integrals, where each integral contains only one source of improperness.

(b) Prove that each of the above integrals is convergent, and hence their sum is convergent as well. (Hint: For the integral near zero, use Taylor series expansions of the functions e^{-xt} and e^{-t}, subtract, divide by t, then integrate. For the integral away from zero, use the inequality $t > \epsilon$ to take t outside the integral, then integrate the rest.)

(9) Consider the function $f(x) = \ln(1+x)$, whose domain of definition is $(-1, \infty)$. We will prove that for $x_0 = a$ in $(-1, \infty)$,

$$\ln(1+x) = \ln(1+a) + \frac{1}{1+a}(x-a) - \frac{1}{2(1+a)^2}(x-a)^2 + \frac{1}{3(1+a)^3}(x-a)^3 - \dots$$

for all $|x-a| < 1+a$.

(a) For $a = 0$, integrate the Taylor expansion for $\frac{1}{1+x}$ around $x_0 = 0$ to prove that the above equality is true. Check the endpoints to make sure you have the right interval of convergence.

(b) Prove that the series

$$\ln(1+a) + \frac{1}{1+a}(x-a) - \frac{1}{2(1+a)^2}(x-a)^2 + \frac{1}{3(1+a)^3}(x-a)^3 - \dots$$

is the Taylor series expansion for the function $\ln(1+x)$ expanded around $x_0 = a$. (Hint: Just plug $f(x) = \ln(1+x)$ and $x_0 = a$ into the formula for the Taylor series.)

(c) Find the interval of convergence of the Taylor series in part (b). (Hint: easy using the n-th root limit.)

(d) Use the Lagrange form of the Taylor remainder $R^n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ to prove that the Taylor expansion in part (b) converges to the function $\ln(1+x)$, whenever $|x-a| < 1+a$. (Hint: Find the pattern for $f^{(n+1)}(c)$, then prove that $\lim_{n\to\infty} R^n(x) = 0$ for $|x-a| < (1+a)$.)

(e) Deduce that the function $\ln(1+x)$ is analytic in its domain of definition.

Famous Analytic Functions

(1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ on $\mathbb{R}$.

(2) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ on $\mathbb{R}$.

(3) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ on $\mathbb{R}$.

(4) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ on $(-1, 1)$.

(5) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$ on $(-1, 1)$ (substitute $-x$ for x in (4)).

(6) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ on $(-1, 1]$ (integrate (5) and check endpoints).

(7) $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ on $[-1, 1]$ (substitute x^2 for x in (5), integrate, then check endpoints) .

In all of the above series, we are expanding around the point $x_0 = 0$, so the series is also called Maclaurin series.