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 To this exam
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MATH 236 (FALL 2014) QUIZ I

THURS SEPT 24, 2014

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Attempt all problems. Box your answers.

- 3 (1) Compute $\int (\ln x)^2 dx$ (Hint: Integration by parts twice). Find the exact value of the average of the function $f(x) = (\ln x)^2$ over the interval $1 \leq x \leq 4$.

$$\begin{aligned}
 & \int (\ln x)^2 dx \quad u = (\ln x)^2 \quad u' = 2 \ln x \frac{1}{x} \quad v = x \quad \int u v' dx = - \int u v dx + u v \\
 & = - \int 2 \ln x \frac{1}{x} x dx + x (\ln x)^2 \\
 & = - 2 \int \ln x dx + x (\ln x)^2 \quad u = \ln x \quad u' = \frac{1}{x} \quad v = x \\
 & = - 2 \left[- \int \frac{1}{x} x dx + x \ln x \right] + x (\ln x)^2 \\
 & = - 2 \left[- \int dx + x \ln x \right] + x (\ln x)^2 \\
 & = 2x - 2x \ln x + x (\ln x)^2 + C
 \end{aligned}$$

$$\begin{aligned}
 \text{average value} &= \frac{1}{4-1} \int_1^4 (\ln x)^2 dx = \frac{1}{3} \left[2x - 2x \ln x + x (\ln x)^2 \right]_1^4 \\
 &= \frac{1}{3} [8 - 8 \ln 4 + 4 (\ln 4)^2] - \frac{1}{3} [2 - 0 + 0] \\
 &= \frac{1}{3} [8 - 16 \ln 2 + 16 (\ln 2)^2] - \frac{2}{3}
 \end{aligned}$$

$$\approx 0.8656$$

(2) Find the following:

$$\text{Q} \quad (a) \int (\sin x \sqrt{\cos x})^3 dx$$

$$\begin{aligned}
 &= \int \sin^3 x \cos^{3/2} x \, dx \quad \text{let } u = \cos x \\
 &= + \int \sin^3 x \cos^{3/2} x \underbrace{\sin x \, dx}_{-du} \\
 &= \int (1 - \cos^2 x) \cos^{3/2} x \sin x \, dx \\
 &= - \int (-u^2) u^{3/2} \, du \\
 &= - \int u^{5/2} - u^{7/2} \, du \\
 &= -\frac{2}{5} u^{5/2} + \frac{2}{9} u^{9/2} + C \\
 &= -\frac{2}{5} \cos^{5/2} x + \frac{2}{9} \cos^{9/2} x + C
 \end{aligned}$$

$$\text{Q} \quad (b) \frac{d}{dx} \int_1^{\tan x} \frac{1}{t^2 \sqrt{t^2+1}} dt.$$

fund. thm. of calc. $\frac{d}{dx} \int_a^{u(x)} f(t) dt = f(u(x)) u'(x)$
& chain rule

$$= \frac{1}{\tan^2 x \sqrt{1+\tan^2 x}} \sec^2 x$$

$$= \frac{1}{\tan^2 x \sqrt{\sec^2 x}} = \frac{|\sec x|}{\tan^2 x}$$

2 (c) $\int \frac{x^3+4x^2-21}{x^2+6x+10} dx.$

$$= \int \left(\frac{x-2}{x^2+6x+10} + \frac{7x+1}{x^2+6x+10} \right) dx$$

$$\begin{array}{r} x-2 \\ \hline x^2+6x+10 \end{array} \overbrace{\begin{array}{r} x^3+4x^2-21 \\ - (x^3+6x^2+10x) \\ \hline -2x^2-10x-21 \\ - (-2x^2-12x-20) \\ \hline 2x-1 \end{array}}^{(\text{degree up}) \quad (\text{degree down})}$$

$$= \frac{x^2-2x}{2} + \int \frac{2x+6-6-1}{x^2+6x+10} dx$$

$$= \frac{x^2-2x}{2} + \int \frac{2x+6}{x^2+6x+10} dx - 7 \int \frac{dx}{x^2+6x+10}$$

$$= \frac{x^2-2x}{2} + \int \frac{du}{u} - 7 \int \frac{dx}{(x+3)^2+1}$$

$$= \frac{x^2-2x}{2} + \ln|u| - 7 \tan^{-1}(x+3) + C$$

$$= \frac{x^2-2x}{2} + \ln|x^2+6x+10| - 7 \tan^{-1}(x+3) + C$$

$$\int_1^3 \frac{1}{x^2(x+1)} dx = \int_1^3 \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} dx$$

$$\frac{1}{x^2(x+1)} = \frac{A(x+1)x + B(x+1) + Cx^2}{x^2(x+1)}$$

$$x=0 \Rightarrow 1=B$$

$$x=-1 \Rightarrow 1=C$$

$$x=1 \Rightarrow 1=2A+2B+C$$

$$\Rightarrow 1=2A+2+1$$

$$\Rightarrow A=-1$$

$$\begin{aligned} \Rightarrow \int_1^3 \frac{1}{x^2(x+1)} dx &= \int_1^3 -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} dx \\ &= \left[-\ln|x| - \frac{1}{x} + \ln|x+1| \right]_1^3 \\ &= -\ln 3 - \frac{1}{3} + \ln 4 + 0 + 1 - \ln 2 \\ &\approx 0.261 \end{aligned}$$

$$\left(= \frac{2}{3} + \ln 4 - \ln 3 - \ln 2 \right) \approx 0.261$$

- (3) Frank is evaluating electric motors to drive automated mixing for some waste tanks that he must maintain. One pump is advertised to have a probability that follows the exponential distribution

$$f(t) = 0.31e^{-0.31t},$$

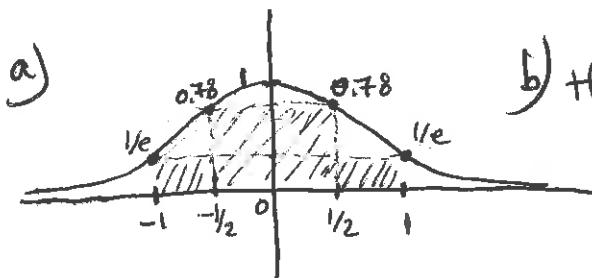
where the time $t > 0$ is measured in years. Frank knows that the expected time of failure for something following this distribution is

$$\int_0^\infty t f(t) dt.$$

How long can he expect one of these pumps to last?

$$\begin{aligned} \int_0^\infty 0.31 t e^{-0.31t} dt &= 0.31 \int_0^\infty t e^{-0.31t} dt = 0.31 \lim_{a \rightarrow \infty} \int_0^a t e^{-0.31t} dt \\ &\quad \text{derivative} \quad \text{integral} \\ &\quad \downarrow \quad \downarrow \\ &= 0.31 \lim_{a \rightarrow \infty} \left[-\frac{1}{0.31} t e^{-0.31t} - \frac{1}{0.31^2} e^{-0.31t} \right]_0^a \\ &\quad (e^{-0.31a} \rightarrow 0 \text{ as } a \rightarrow \infty) \\ &= 0.31 \left[\frac{1}{0.31^2} \right] = \frac{1}{0.31} \approx 3.23 \text{ years} \end{aligned}$$

- (4) (a) Plot the graph of the function e^{-x^2} .
 (b) Graphically, explain why $\int_{-1}^1 e^{-x^2} dx > 1$.
 (c) Give an estimate for $\int_1^\infty e^{-x^2} dx$ by comparison to the integral $\int_1^\infty xe^{-x^2} dx$.
 5 (d) Use parts (b), (c) and your graph in part (a) to estimate $\int_{-\infty}^\infty e^{-x^2} dx$.
 (e) Find the actual value $\int_{-\infty}^\infty e^{-x^2} dx$ using a calculator or your smart phone. (In Calc III, you will be able to evaluate this integral).



b) the shaded rectangles have area
 $1 \times 0.78 + \frac{1}{2} \times \frac{1}{e} + \frac{1}{2} \times \frac{1}{e} \approx 1.14 > 1$

c) since $x \geq 1$ on the interval $[1, \infty)$

then $xe^{-x^2} > e^{-x^2}$

$$\Rightarrow \underbrace{\int_1^\infty xe^{-x^2} dx}_{\text{let } u = x^2} > \int_1^\infty e^{-x^2} dx$$

$$\text{let } u = x^2$$

$$\Rightarrow du = 2x dx$$

$$= \int_1^\infty e^{-u} \frac{1}{2} du$$

$$= -\frac{1}{2} e^{-u} \Big|_1^\infty$$

$$= -\frac{1}{2} e^{-1} = \frac{1}{2e}$$

$$\Rightarrow \frac{1}{2e} > \int_1^\infty e^{-x^2} dx$$

↑
shaded area

d) from a, b & c we know that

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \int_{-\infty}^{-1} e^{-x^2} dx + \int_{-1}^1 e^{-x^2} dx + \int_1^{+\infty} e^{-x^2} dx$$

$$= 2 \int_1^\infty e^{-x^2} dx + \int_{-1}^1 e^{-x^2} dx$$

$$\approx \frac{2}{2e} + 1.14 \approx 1.5$$

e) $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \approx 1.772$

very very rough estimate $\Rightarrow \approx$

- (5) For which values of p does the improper integral $\int_0^1 x^p \sin \frac{1}{x} dx$ converge? (Hint: It may be easier to use the substitution $u = \frac{1}{x}$, then $du = \dots$. Then compare your new integral (with new boundaries) to an integral that you know. Make sure that your answer covers all possible values of p).

$$\text{Let } u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx = -u^2 dx$$

$$\Rightarrow dx = -\frac{1}{u^2} du$$

$$\int_0^1 x^p \sin \frac{1}{x} dx = \int_{\infty}^1 \frac{1}{u^p} \sin u \cdot -\frac{1}{u^2} du$$

$$= - \int_{\infty}^1 \frac{1}{u^{p+2}} \sin u du$$

$$= \int_1^{\infty} \frac{1}{u^{p+2}} \sin u du$$

$$\left| \int_1^{\infty} \frac{\sin u}{u^{p+2}} du \right| \leq \int_1^{\infty} \frac{|\sin u|}{u^{p+2}} du \quad \text{but } |\sin u| \leq 1$$

$$\leq \int_1^{\infty} \frac{1}{u^{p+2}} du \quad \text{this integral converges whenever } p+2 > 1$$

Note that:
 (We only proved that if $p > -1$ then

$$\int_0^1 x^p \sin \frac{1}{x} dx \text{ converges}$$

$$\Rightarrow p > -1$$

(We need to prove that if $p \leq -1$ then the integral would not converge, but I will give full pts for the above argument)