MATH 236 (FALL 2014) SEQUENCES AND SERIES OF NUMBERS

CHAPTER 7 SUMMARY

1. Sequences of numbers $\{a_n\}$

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots, a_N, a_{N+1}, \ldots\}$$

We are interested in the *tail* of the sequence. What are *infinitely* many terms doing? Accumulating in one place (parent sequence and all its subsequences converging to the same limit e.g. $\{\frac{2n^2}{(n+1)(n-7)}\}$)? Accumulating in more than one place (has subsequences converging to different limits e.g. $\{(-1)^n\}$, $\{\sin(\frac{n\pi}{3})\}$)? Going to one of the infinities (all subsequences blowing up at one of the infinities e.g. $\{\frac{3^n}{2}\}$)? Going to infinity and accumulating in some other place (one subsequence blowing up and another one converging e.g. $\{(-1)^n n + n\}$)? *Etc....*

(1) **Concepts:** We have the following concepts and they are related: Convergence, Divergence, Non-convergence, Boundedness, Monotonicity, Subsequences.

- **Convergence:** All but finitely many terms of (a_n) are within an ϵ distance of L (so the whole tail of the sequence after a certain cutoff N, $\{a_N, a_{N+1}, a_{N+2}, \ldots\}$ lives within an ϵ-neighborhood of L).
  Formal definition: $a_n \to L$ as $n \to \infty$ iff: given $\epsilon > 0$, there exists a cutoff N such that
  $$\text{for all } n > N,\ \ |a_n - L| < \epsilon.$$
- **Divergence:** The tail of the sequence crosses any positive barrier M, no matter how large it is.
  Formal definition: $a_n \to \infty$ as $n \to \infty$ iff: given $M > 0$, there exists an N such that
  $$\text{for all } n > N,\ \ a_n > M.$$
  Similarly for $a_n \to -\infty$ (the tail of the sequence crosses any negative barrier M, no matter how large and negative it is) iff given $M < 0$, there exists and N such that
  $$\text{for all } n > N,\ \ , a_n < M.$$
- **Boundedness:** The whole sequence lives within bounds $-M$ and M.
  Formal definition: (a_n) is bounded iff there exists an M such that
  $$|a_n| \leq M \ \text{ for all } n \in N.$$
  You may want to find the *least upper bound* and the *greatest lower bound* for any sequence that you encounter. These could be infinite. They also may be different than the limit of the sequence.
- **Subsequence:** (a_{n_k}) is a subsequence of (a_n). $a_{n_k} \to a$ as $k \to \infty$ iff infinitely many terms of the parent sequence (a_n) are within an ϵ distance of a. So the terms of the parent sequence visit an ϵ neighborhood of a (or a itself) infinitely many times. That is, a is an *accumulation point* of the parent sequence (a_n), or a is a *subsequential limit.* Note that there could be other accumulation points.
- **Monotone sequence:** (a_n) is monotone increasing iff $a_n \leq a_{n+1}$ for all $n \in N$. (a_n) is monotone decreasing iff $a_n \geq a_{n+1}$ for all $n \in N$. A monotone sequence is either convergent (goes to a unique finite limit) or divergent (goes to one of the infinities).
  Testing for a monotone sequence

(a) Easiest way, but not a proof: Replace n with x, plot the function $a(x)$, and see whether the graph of the function $a(x)$ is decreasing for $x \geq$ where the index n starts.
(b) Difference: check whether $a_{n+1} - a_n$ is > 0 or < 0 for all $n >$ some index.
(c) Ratio: if none of the a_n's are zero, check whether $\frac{a_{n+1}}{a_n}$ is ≥ 1 or ≤ 1 for all $n >$ some index.
(d) Derivative: Replace n with x and check whether $a'(x)$ is ≥ 0 or ≤ 0 for $x \geq$ some value, of course given that the function $a(x)$ is differentiable.

(2) **Relationships between the above concepts**
(a) Convergent sequence:
- Every convergent sequence is bounded.
- All subsequences of a convergent sequence converge to the same limit. Conversely, if all subsequences converge to the same limit, then the parent sequence itself converges to that limit.

(b) Bounded sequence:
- Every bounded sequence has a convergent subsequence (infinitely many points living within a restricted space have to accumulate in one place or in more than one place, e.g. $\{(-1)^n\}$).
- A bounded sequence does not have to be convergent (example: $\{(-1)^n\}$, $\{\cos \frac{n\pi}{6}\}$).

(c) Monotone sequence:
- Every monotone bounded sequence is convergent (monotone convergence theorem).
- If a monotone sequence is unbounded, then it is divergent (its limit is one of the infinities).

(So the limit of a monotone sequence is either a unique finite number or one of the infinities).
(d) Divergent sequence: is unbounded. All of its subsequences diverge as well.
(e) Unbounded sequence: has a divergent subsequence (which is causing it to become unbounded).

(3) **Newton's method for finding the roots of a function** $f(x)$. The roots are the points x^* where $f(x^*) = 0$, or the points x^* where the graph of $f(x)$ meets the x-axis. Most equations $f(x) = 0$ are hard to solve analytically for x, e.g. $e^x - (\sin x)^2 = 0$, so we need a numerical method to find the roots. Newton's method gives a sequence of numbers $\{x_n\}$ that converges (under certain conditions) to a root of $f(x)$:
(a) Start with a given initial *guess* x_0.
(b) Iterate using the recurrence relation: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ (plug in $k = 0, 1, 2, \ldots$ to form a sequence of numbers).
(c) Stop when $|x_{k+1} - x_k| <$ a given tolerance, e.g. 10^{-3}. (So check the difference between two consecutive iterates, and stop when that difference is less than your tolerance.)
(d) You can plug in your last answer (say you stopped at x_7) into $f(x)$, and check that $f(x_7) \approx 0$ to make sure that your sequence indeed converged to a root of $f(x)$.

(4) **Fixed point iteration for finding a fixed point of a function (**$x = g(x)$**).** We did this in Quiz II.

2. Series of Numbers $\sum_{n=1}^{\infty} a_n$

Series of numbers $\sum_{n=1}^{\infty} a_n$ is equal to $\lim N \to \infty \sum_{n=1}^{N} a_n = \lim_{N\to\infty}(a_1 + a_2 + a_3 + \ldots + a_N)$. We basically want to add all the terms of the sequence together: the *tail* of the sequence needs to be converging to zero *fast enough* in order to guarantee that the sum of the *infinitely* many terms in the tail will not blow up (e.g., if the tail accumulates near $\frac{1}{e}$ and not zero, when you add up these infinitely many numbers close to $\frac{1}{e}$, their sum will be infinite!).

For a given series, we ask: **Does it converge? Diverge? No limit? If it converges, do we know what the answer is (for geometric and telescoping series, we know the answer), or can we estimate it (yes for series where the integral test applies, or for alternating series)**?

Thought Process:

A) Is the series famous?
- p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Converges if $p > 1$ and diverges otherwise. ($\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series* and it diverges.)
- Geometric: $\sum_{n=\text{first index}}^{\infty} r^n$ converges to $\frac{r^{\text{first index}}}{1-r}$ if $|r| < 1$, and diverges if $r \geq 1$.
- Telescoping: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(b_n - b_{n+1}) = \lim_{N\to\infty} b_1 - b_{N+1}$ (we compute a limit instead of an infinite sum).

A') Is the series alternating ($\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$) or does it possess positive and negative terms?
- If it is alternating, does the alternating test apply? **Alternating test**: Series converges if a_n's are positive, monotone decreasing, and $\lim_{n\to\infty} a_n = 0$. (Note that the alternating test only gives a conclusion when it applies, if the alternating test does not apply, you need to think of something else.)
- Is this series absolutely convergent? (Study the *new* series $\sum_{n=1}^{\infty} |\text{original terms}|$, using A, B, C, D, E thought process). If it is absolutely convergent then it is convergent.
- Is it only conditionally convergent? (A conditionally convergent series need not converge to the same limit if its terms are rearranged, in fact, in need not stay convergent at all if its terms are rearranged. So its convergence to a certain limit is *conditioned* on the particular order of its positive and negative terms.)
- Maybe the series is not exactly alternating but it has a part that 'serves like' the alternator $(-1)^n$, for example, the series $\sum_{n=1}^{\infty} \frac{\sin\sqrt{n}}{n}$ or $\sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2}}$. In this case, your intuition about what would happen with the similar alternating series (with an actual alternator $(-1)^n$) should be correct. You may want to use the fact that $|\cos(\text{anything})| \leq 1$ and $|\sin(\text{anything})| \leq 1$ and do an actual comparison with a p-series (this will work for $\sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2}}$ but not for $\sum_{n=1}^{\infty} \frac{\sin\sqrt{n}}{n}$: this one is like $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally and not absolutely). The rigorous way to justify your 'serve like' intuition is to use Dirichlet test, which is beyond the scope of Math 236.

B) Is $\lim_{n\to\infty} a_n \neq 0$? If the limit of the terms is nonzero then the series does not converge. If that limit is zero then it could go either way, so keep going.

C) Does the integral test apply? **Integral test**: Replace n by x and plot (or study) the function $a(x)$:
   (a) Is $a(x) > 0$ (graph above x-axis)?
   (b) Is $a(x)$ decreasing for $x > s$ for some value s?
   (c) Is $a(x)$ continuous?

   If the answer is yes to the above three conditions (show a graph for that, or show a proof), then $\sum_{n=1}^{\infty} a_n$ and $\int_s^{\infty} a(x)dx$ behave in the same way. They converge together or diverge together.

D) Is your series "like $\sim$" a series that you know? Justify your "$\sim$" with either an **actual comparison** or a **limit comparison** (here series must have positive terms).
   - **Actual comparison:** If $0 \le a_n \le b_n$ for some $n > N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well. If $0 \le b_n \le a_n$ for some $n > N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.
   - **Limit comparison:**
     (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ finite, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
     (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.
     (c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ also diverges.

E) Try the **Ratio** or the **Root** tests for series that seem to have lots of powers or some sort of factorials in them (series must have positive terms). So compute either $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty}(a_n)^{1/n} = L$ (depending on which limit is easier to compute). Compare your limit to 1: If $L < 1$ the series converges, if $L > 1$ the series diverges, and if $L = 1$ then there is no conclusion (the ratio and root limits for both of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are equal to 1).

Careful: Integral test, Ratio and Root tests, actual and limit comparison tests, only work for series of *nonnegative* terms. Series we encounter with negative terms could be: geometric, telescoping, alternating, or just other series with positive and negative terms (caused by sin's and cos's for example, or just anything that takes positive and negative values.) You may want to check absolute convergence first for series with negative terms that are not telescoping, geometric, or non-alternating.

3. Remarks, Helpful Tips and Limits

(1) **Computing and estimating the limits of series**:
   - The series that we know their exact answer (limit) if they are convergent: Geometric series and Telescoping series.
   - The series that we can *estimate* their answer if they are convergent:
     (a) Series for which the integral test applies: We can estimate the series by the finite sum, and the error is bounded by the integral.

$$a_1 + a_2 + \ldots + a_m \le \sum_{n=1}^{\infty} a_n \le a_1 + a_2 + \ldots + a_m + \int_m^{\infty} a(x)dx$$

     and the remainder (error due to truncating after m terms) is bounded by

$$0 \le a_{m+1} + a_{m+2} + \ldots \le \int_m^{\infty} a(x)dx.$$

     The more terms you use the smaller this error will be because the series is convergent. Note that the boundary of the integral here starts at the point where you truncated, and that is different than the boundary you used while applying the integral test.

     (b) Alternating series for which the alternating series test applies:

$$|\sum_{n=1}^{\infty}(-1)^{n \text{ or } (n+1)} a_n - (a_1 + a_2 + \ldots + a_m)| < a_{m+1}$$

     and the sign of the difference is the sign of the term a_{m+1}.

In later math courses, after studying complex analysis, Fourier series and more, we can compute exact sums of much more series of numbers.

(2) **The dominance relationship**: For n large enough, logarithms are dominated by powers dominated by exponentials dominated by factorials. Let $s > 0$ and $b > 1$, then for n large enough, we have

$$\ln n << n^s << b^n << n! << n^n$$

so for example $\lim_{n\to\infty} \frac{n^{2/3}}{\ln n} = \infty$, $\lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} (n!)^{1/n} = \infty$ and $\lim_{n\to\infty} \frac{e^{(n+1)}}{n!} = 0$.

(3) **Helpful limits to always remember**

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.
(b) $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$.
(c) $\lim_{n\to\infty} b^{1/n} = \lim_{n\to\infty} \sqrt[n]{b} = 1$ for $b > 0$.
(d) $\lim_{n\to\infty} \frac{b^n}{n!} = 0$. (Dominance of factorials over powers.)
(e) $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$.
(f) $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} = 1$.
(g) $\lim_{n\to\infty} \left(1 + \frac{s}{n}\right)^n = e^s$.
(h) $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$
(i) $\lim_{m\to 0} \frac{\sin m}{m} = 1$.
(j) $\lim_{n\to\infty} \frac{\cos n}{n} = 0$ (or any bounded quantity divided by infinity).
(k) Finite Geometric sum: $1 + r + r^2 + ... + r^n = \frac{1-r^{n+1}}{1-r}$ for $r \neq 1$.
Geometric series:

$$\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} (1 + r + r^2 + ... + r^n) = \frac{r^0 \text{ or } r^{\text{first index}}}{1-r}$$

for $|r| < 1$, hence,

$$a + ar + ar^2 + \ldots = \frac{a}{1-r} = \frac{\text{first term}}{1-r}$$

(4) **Helpful Tips**

(a) $\sin x \sim x$ for small x, so $\sin \frac{1}{n} \sim \frac{1}{n}$ for large n. (Justification: $\lim_{m\to 0} \frac{\sin m}{m} = 1$, or later, Taylor series expansion). E.g. $\sum_{n=1}^{\infty} \sin^2 \frac{1}{n} \sim \sum_{n=1}^{\infty} \frac{1}{n^2}$. Justify the $\sim$ by a limit comparison.
(b) $\sin n$ or $\cos \sqrt{n}$ and similar terms 'serve like' alternators $(-1)^n$ in series like $\sum_{n=1}^{\infty} \frac{\sin \sqrt{n}}{n}$ *etc...*
(c) We can always bound sin's and cos's by 1: $|\sin(\text{anything})| \leq 1$ and $|\cos(\text{anything})| \leq 1$.

(5) **Applications**: We saw fractals, compounded interests, radioactive decay, bouncing balls, and others.