Math 236 Calculus II (Fall 2014)
Taylor Polynomial, Taylor Remainder, Taylor Series

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For a function $f(x)$ we have three objects that are related:

1. If $f(x)$ is smooth (infinitely differentiable) in a neighborhood of a, we can construct its **Taylor series expanded about $x_0 = a$**:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \ldots.$$

2. If $f(x)$ has at least n-derivatives in a neighborhood of a we can construct its **Taylor polynomial of degree n about $x_0 = a$**:

$$P^n_{x_0=a}(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

   Taylor polynomial terminates after n-terms but Taylor series keeps going.

3. **The Taylor remainder** is the difference between $f(x)$ and its Taylor polynomial:

$$\begin{aligned} R^n_{x_0=a}(x) &= f(x) - P^n_{x_0=a}(x) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ (Lagrange's form of the remainder, very useful!)} \\ &= \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt \text{ (Integral form of the remainder)} \end{aligned}$$

   provided that $f^{(n+1)}$ (the $(n+1)$'th derivative of f) is continuous on an interval I containing a and x, and c is a point between a and x.

1 Goals

Since polynomials are easy functions to deal with, and to evaluate, we want to know:

- How well can we approximate a function (that has enough derivatives) near a point x_0 by its Taylor polynomial expanded around that point (finite number of terms of its Taylor series)? How much error are we committing (Taylor remainder) by doing that?

- When does a smooth function and its Taylor series agree? (They agree on the interval where Taylor remainder $\to$ zero as $n \to \infty$. Cases: they may agree only at the point of expansion $x_0 = a$, the whole real line $\mathbb{R}$, or a finite interval). A function which agrees with its Taylor expansion on an interval is called an **analytic function** on that interval. Rephrase the question: **Can a function be represented by its Taylor series around a point, near that point?** Answer: Yes if the Taylor remainder goes to zero in a neighborhood of that point, and that makes the function analytic in that neighborhood.

2 Notes

1. Even though the Taylor series is constructed using the function $f(x)$, it **need not** converge to $f(x)$. In fact, some Taylor series agree with the function $f(x)$ nowhere except at the point x_0 (for example, e^{-1/x^2} and it's Taylor series around zero agree nowhere except at the point zero). Those functions which agree with their Taylor series expansion on an interval are called analytic on that interval. So, whenever you write a function equal to its Taylor series expansion, you need

to **specify the interval on which this equality is valid.**

2. Thus, **an analytic function is smooth**, **but a smooth function need not be analytic.**

   Later math classes: This is very different from complex valued functions, where analytic functions and infinitely differentiable functions are the same.

3. $\lim_{n\to\infty}$ of Taylor polynomial=Taylor series.

4. Taylor series is a power series (has the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ where $a_n = \frac{f^{(n)}(a)}{n!}$). It converges at $x = a$ to $f(a)$ (so Taylor series always agrees with $f(x)$ at the point of expansion, and the question is how far can we push beyond the point of expansion and still have the function agreeing with its Taylor series, or well approximated by a Taylor polynomial of certain degree). Everything we know about power series (see Section 3 below) applies to Taylor series: we can find its radius of convergence, interval of convergence, and type of convergence (absolute within the interval, check endpoints individually). Again, the Taylor series could be converging to something completely different than the function $f(x)$ away from the point of expansion x_0. It could also be not converging at all away from x_0.

5. **Term by Term Differentiation and Integration**

   As long as we are within the interval of convergence of a power series, we can differentiate it or integrate it term by term as many times as we want. The new series that we obtain has the same interval of convergence as the one we started with, except possibly at the endpoints, which we can check individually (plug in the end points into the new power series that you obtained through differentiating or integrating).

   We can use this fact to construct Taylor series of analytic functions using Taylor series of other analytic functions, without having to go through the pain of differentiating the function many times, getting the pattern, then finding the Taylor remainder and checking whether it goes to zero as $n \to \infty$.

   e.g. Prove that the function $\tan^{-1}(x)$ is analytic on the interval $[-1, 1]$ and write down its Taylor expansion around a point in that interval.

   Soln Integrate the Taylor series for the function $\frac{1}{1+x^2}$ obtained by substituting $(-x^2)$ for x in the series for $\frac{1}{1-x}$ on the interval $(-1, 1)$. Then check the endpoints -1 and 1 in the new series individually (see (7) in Section 5 below).

6. **So how to know whether a function is analytic** on an interval? We learned two methods:

   (a) Find the Taylor remainder $R^n(x)$ (compute derivatives of f until you get the pattern), then find $\lim_{n\to\infty} R^n(x)$, if the limit for x in a certain interval is zero, then the function is analytic on that interval.

   (b) If you can derive your function using another analytic function that you know by substitution, differentiation, or integration, then that is much easier than part (a) (for example, the $\tan^{-1}(x)$ function above).

3 Power Series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$

A power series always converges to a_0 at $x = x_0$. The question is how far can we push beyond x_0 and still have the power series converge?

1. Compute either

   $$\lim_{n\to\infty} \sqrt[n]{|a_n|}|x-x_0| \text{ or } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x-x_0|.$$

   Do not forget the absolute values.

2. The power series converges absolutely at the interval where the above limit is < 1. Solve the inequality for x to find the interval of convergence. If the limit is a number that is always less than 1, then the series is absolutely convergent everywhere on $\mathbb{R}$, and the radius of convergence is infinity.

3. We have to check the endpoints of the interval alone.

   (a) If the series converges absolutely on one endpoint then it also converges absolutely on the other one and no need to check the other one.

   (b) If the series converges conditionally at one endpoint then we need to check the other one (could converge conditionally there or not converge at all).

   (c) If the series does not converge at one endpoint then we need to check the other one (could converge conditionally there or not converge at all).

4. The radius of convergence R is how far we could push beyond x_0 on either side (half of the width of the interval of convergence) and still have the series converge. The radius could be 0 and when the series converges nowhere except at the point x_0, finite when the series converges on a finite interval $(x_0 - R, x_0 + R)$ with endpoints included or excluded, or infinite when the series converges everywhere on $\mathbb{R}$.

5. A convergent power series defines a function on its interval of convergence (this function could coincide with a function that we know, or not. For example, the power series $\sum_{n=0}^{\infty} x^n$ coincides with the function $\frac{1}{1-x}$ on the interval $(-1, 1)$ (because it is a geometric series with $r = x$, we know its sum). Note that this power series does not converge outside this interval, while the function $\frac{1}{1-x}$ is perfectly fine outside the interval, except at $x = 1$, where it has a vertical asymptote. Conversely, this makes the function $\frac{1}{1-x}$ analytic on the interval $(-1, 1)$, with $\sum_{n=0}^{\infty} x^n$ its Taylor series around the point $x_0 = 0$.) As long as we are inside the interval of convergence of the power series, we can differentiate it and integrate it term by term. We check the endpoints of the new series individually.

   Hence, a power series defines a continuous, differentiable, and integrable function within its interval of convergence.

4 Summary

You need to know what each of the following means:

1. Power series.
2. Taylor series.
3. Taylor polynomial.
4. Taylor remainder.
5. Smooth function.
6. Analytic function on an interval.

You also need to know

1. How to determine whether a smooth function is analytic on an interval.
2. Examples of analytic functions and intervals on which they are analytic and examples of non-analytic functions.
3. Graphing a function, its Taylor polynomials of various degrees around a point, and identifying graphically the Taylor remainder at various points near the expansion point.
4. Using analytic functions to find the exact sums of series of numbers, for example $\sum_{n=0}^{\infty} \frac{(-1)^n}{n} = \ln(2)$ (plug $x = 1$ in (6) in Section 5 below).

   Conversely approximate numbers like $e^{0.5}$ using finite sum of the Taylor series of the analytic function (put $x = 0.5$ in (1) in Section 5 below, if you use say 7 terms of the series, then the error is the Taylor remainder $R^7(0.5)$ which you have a formula for on page 1).

5 Some famous analytic functions

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ on $\mathbb{R}$.
2. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ on $\mathbb{R}$.
3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ on $\mathbb{R}$.
4. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ on $(-1, 1)$.
5. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$ on $(-1, 1)$ (substitute $-x$ for x in (4)).
6. $\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ on $(-1, 1]$ (integrate (5) and check endpoints).
7. $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ on $[-1, 1]$ (substitute x^2 for x in (5), integrate, then check endpoints).

In all of the above series, we are expanding around the point $x_0 = 0$, so the series is also called Maclaurin series.

6 More on Analytic Functions

A function $f(x)$ is analytic on an interval (a, b) iff its Taylor expansion around each point x_0 in (a, b) converges to $f(x)$ in a neighborhood of x_0. That is, the function $f(x)$ can be represented by its Taylor series expansion near every point of the interval (a, b).

1. Every smooth function (possesses continuous derivatives of all orders) has a Taylor series expansion but this Taylor series need not converge to the function $f(x)$. If it does in a neighborhood of x_0 then the function is analytic in that neighborhood.

2. **Example:** Consider the function $f(x) = \ln(1 + x)$. Its domain of definition is $(-1, \infty)$. Pick a point of expansion $x_0 = a$ in its domain. Construct the Taylor series around that point:

$$\ln(1+a) + \frac{1}{1+a}(x-a) - \frac{1}{2(1+a)^2}(x-a)^2 + \frac{1}{3(1+a)^3}(x-a)^3 - \dots.$$

   This Taylor series converges whenever $|x-a| < 1+a$ (so if $a = 0$, it converges for |x|<1, and if $a = 3$, it converges for $|x - 3| < 4$; the endpoints need to be checked individually.) Now using the Lagrange form of the Taylor remainder $R^n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, we can compute this remainder and prove that $\lim_{n\to\infty} R^n(x) = 0$ for $|x - a| < 1 + a$, and hence the above Taylor series converges to the function $\ln(1 + x)$ in that interval. That is,

$$\ln(1+x) = \ln(1+a) + \frac{1}{1+a}(x-a) - \frac{1}{2(1+a)^2}(x-a)^2 + \frac{1}{3(1+a)^3}(x-a)^3 - \dots \text{ for } |x-a| < 1+a.$$

   Since this argument is valid for any point a in the domain of $\ln(1 + x)$, we have proved that: Near every point $x_0 = a$ (in the neighborhood $|x - a| < 1 + a$) in the domain of $\ln(1 + x)$, this function can be represented by its Taylor series expansion around that point. Hence $\ln(1 + x)$ is analytic on the interval $(-1, \infty)$.
   Note that for $a = 0$, we could have just integrated the Taylor series for the function $\frac{1}{1+x}$ within the interval $(-1, 1)$ (and checked endpoints individually), and gotten the equality

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \text{ on } (-1, 1].$$

   From this, we can only deduce that $\ln(1 + x)$ is analytic on the interval $(-1, 1)$, but not on all of its domain $(-1, \infty)$ as we proved above.

3. Real valued functions which are analytic on their domains are sparse and only certain special functions are. On the contrary, for complex valued functions, differentiable functions are automatically analytic.

7 Exercises

1. (a) Prove that the function $\frac{\sin x}{x}$ is analytic on $\mathbb{R}$ and find its Taylor expansion around $x_0 = 0$. (Hint: Use (2) from section 5 above.)
   (b) Plot $\frac{\sin x}{x}$ and its Taylor polynomial approximations of order 6, 8 and 10, on the interval $(-10, 10)$.
   (c) What must you change if we were to consider the function $\frac{\cos x}{x}$ instead?

2. (We did this in class) Show that the function e^{-1/x^2} and its Taylor series expansion around $x_0 = 0$ disagree everywhere except at $x = 0$. (In fact, the Taylor series around $x_0 =$ is identically zero, while the function e^{-1/x^2} is zero only at $x = 0$.) Plot a figure to illustrate. Is this function analytic anywhere?

3. Consider the improper integral

$$\int_0^\infty \frac{e^{-xt} - e^{-t}}{t} dt,$$

   where $x > 0$.

   (a) Why is this integral improper? Split the integral into the sum of two integrals, where each integral contains only one source of improperness.
   (b) Prove that each of the above integrals is convergent, and hence their sum is convergent as well. (Hint: For the integral near zero, use Taylor series expansions of the functions e^{-xt} and e^{-t}, subtract, divide by t, then integrate. For the integral away from zero, use the inequality $t > \epsilon$ to take t outside the integral, then integrate the rest.)