Math 336 fall 2015

Week 11 Highlights

Numerical methods. Chapter 9 in the book

1st order ODE $y' = f(t,y)$

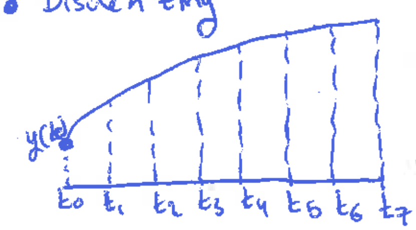
- analytical solns for
  - separable (linear, nonlinear)
  - 1st order linear (I.F.)
  - exact (linear, nonlinear)

- graphical approximate solns
  - slope field (linear, nonlinear)
  - critical pts & stability

- Numerical (linear, nonlinear, single eqn, system of ODEs) ← loss of information due to discretizing
  ↖ So there will be error term
  - matlab (so far used ODE45)
  - Euler's method
  - Improved Euler
  - Runge kutta

- How good is a numerical soln: depends on the prob & the corresponding numerical method used. Some method work better for certain probs & not so well for others.

- Discretizing



discrete → continuum :
a limiting process as $h \to 0$
h = mesh size

- ERROR:
  - → error at each step
  - → cummulative error

  } from the numerical method
  - → roundoff error } from the computer

- smaller stepsize
  - better accuracy
  - more computational cost
  - too small stepsize, roundoff error plays role!

- exact soln: $y(t)$
- discretized exact soln: $(y(t_0), y(t_1), \ldots, y(t_n))$
  (for many probs, we do not know what the exact soln is, however, probs for which we do know, are usually great to TEST the numerical method).
- Numerical soln: $(\tilde{y}(t_0), \tilde{y}(t_1), \ldots, \tilde{y}(t_n))$
- Good numerical method
  - small error $|y - \tilde{y}|$
  - computational nonexpensive

Euler's Method: approximate the graph of the exact soln with a sequence of tg line approximations.
(tg line at each step is the tg to the solution curve at (x_n, y_n) not at $(x_n, y(x_n))$, & hence the error)
y_n ← approximate, $y(x_n)$ ← exact

Euler's scheme: $y_{n+1} = y_n + h f(t_n, y_n)$, $n = 0, 1, 2, \ldots$ } to numerically approximate soln of
$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \\ t \in [t_0, t^*] \end{cases}$$

Error: each step error $\varepsilon_n = |\underset{\text{exact}}{y(x_n)} - \underset{\text{approximate}}{y_n}| \le C h^2$, $C = \frac{|y''(\xi)|}{2}$

cummulative error $\varepsilon = \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon^* \le M h$

so ε_n is of $O(h^2)$ & ε is of $O(h)$

Catastrophic failure of Euler's method: eg $y' = (y-1)^2$ } ($y=1$ unstable critical pt for the ODE)

Refinements on Euler's method can yield schemes in which the numerical soln converges to the true soln much faster as the step size decreases. (Recall, because of $e_j \sim O(h^2)$ & $E \sim O(h)$, the approximate soln is, in theory, guaranteed to converge to the [redacted] exact soln as $h \to 0$. It will converge faster for methods with "higher accuracy", or $E \sim O(h^{\text{higher power}})$.

Improved Euler & Runge Kutta methods: $f(t_n, y_n)$ in Euler's method (1 evaluation of $f(t,y)$ at the pt (t_n, y_n) at each step),

is replaced by carefully chosen "weighted averages" of $f(t,y)$ evaluated at carefully chosen pts near (t_n, y_n).

This leads to more accuracy, but obviously more computations at each step.

- **Improved Euler method:** 2 evaluations of $f(t,y)$ at (t_n, y_n) and (t_{n+1}, u_{n+1}) at each step.

Improved Euler's scheme:

$$\begin{cases} u_{n+1} = y_n + h\, f(t_n, y_n) \quad \text{(same as Euler)} \\ y_{n+1} = y_n + \underbrace{\tfrac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, u_{n+1})\right]}_{\text{average slope at } (x_n, y_n) \text{ \& at } (x_n,\ \text{approximate } y_{n+1} \text{ guessed by Euler})} \end{cases}$$

or

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_{n+1}, u_{n+1})$$
$$y_{n+1} = y_n + \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2$$

cummulutative error: $E \sim O(h^2)$

- **Runge Kutta Method:** 4 evaluations of $f(t,y)$ at (t_n, y_n), $(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$, $(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$, $(t_{n+1}, y_n + hk_3)$

Runge Kutta scheme

$$\begin{cases} k_1 = f(x_n, y_n) \\ k_2 = f(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hk_1) \\ k_3 = f(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hk_2) \\ k_4 = f(x_{n+1},\ y_n + hk_3) \\ y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

$E \sim O(h^4)$

You need to know how to apply the above numerical methods to 1st order ODEs (linear, nonlinear, const coeff's, nonconst coeff's).

The Error:

- Error in Euler's method is of order h

$|y_n - y(x_n)| \leq Ch$ for each $n = 1, 2, 3, \ldots, k$

Change the size of h, the error changes linearly with h:

say new $h = \frac{\text{old } h}{10} \Rightarrow$ get $\frac{1}{10}$ max error so ten times more accuracy!

Say you don't know the exact soln:

→ Apply Euler's method with some h.

→ repeat with $\frac{h}{2}, \frac{h}{4}, \ldots$

→ continue until results at one stage agree to an appropriate number of significant digits. Then the approximate values at this stage are considered likely to be accurate up to the indicated no of significant digits.