Math 336 Ordinary Differential Equations Final Exam

Fall 2015

Name:

Box your answers.

1 Solve the following ODEs or IVPs

If the ODEs admit complex solutions, write BOTH the complex solutions and the real solutions.

1. $x^2(1+y^2) + 2y\frac{dy}{dx} = 0$ with $y(0) = 1$.

2. $y'' - y' - 6y = 3t + 2$.

3. *Power series method*: $(1 + x^2)y'' + 2xy' - 2y = 0$. Can you express your series solution in terms of elementary functions?

2 Systems

1. The matrix $A = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}$ has an eigenvalue $\lambda_1 = 2i$ and a corresponding eigenvector $\underline{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$. Find the real solution of the system $\underline{w}' = A\underline{w}$. Make an accurate phase portrait.

2. *Ecological model*: Analyze the type and stability of the critical points of the following predator-prey ecological system, then plot a phase portrait showing all interactions and the separatrices if any. Based on your phase portrait, make future predictions for the behavior of the two species.

$$\begin{cases} x'(t) = 7x - x^2 - 2xy \\ y'(t) = y - y^2 + 2xy. \end{cases}$$

3 Solve the following to the best of your knowledge

1. A simple model for air resistance is that the force is proportional to and opposing velocity such that the velocity of a falling object as a function of time satisfies the following differential equation,

$$mv' = -\gamma v - mg,$$

where $m > 0$ is the object's mass, $\gamma > 0$ is the resistance coefficient, and $g > 0$ is the gravitational acceleration constant.

   (a) Solve the initial value problem with $v(0) = 0$.

   (b) Determine the limiting velocity.

2. (a) Define resonance.

   (b) A mass-spring system can be modeled using the following ODE,

$$mx'' + cx' + kx = f(t),$$

   where m is the mass, c is the resistance constant, k is the spring constant, and $f(t)$ is an external force applied to the system. Consider a cart whose weight is $128lb$ that is attached to a wall by a spring with spring constant $k = 64lb/ft$. Initially, the cart is pulled $0.5ft$ in the direction away from the wall and released with no initial velocity. Simultaneously, a periodic external force $f(t) = 32\sin(4t)$ is applied to the cart. Assuming that there is no air resistance, find the position of the cart at any time t. Plot your solution, *illustrating the resonance phenomenon.* (Note that to obtain the mass of the cart, you need to divide its weight by the gravity $g = 32ft/sec^2$.)

4 Theory

1. State Picard's existence and uniqueness theorem for a first order ODE

$$\frac{dy}{dx} = F(x, y).$$

2. Use Picard's iteration $y_{j+1}(x) = y_0 + \int_{x_0}^{x} F(s, y_j(s))ds$ to solve the ODE $y' = 2y$ with $y(0) = 1$. (Note that this IVP is easy to solve, and the solution is $y(x) = e^{2x}$, but Picard's iteration with $x_0 = 0$ and $y_0(x) = y_0 = 1$ gives you a sequence of functions that converges to e^{2x}.)