Math 336 Ordinary Differential Equations Written Assignment 4

Power Series Methods

Power series methods (PSMs) are useful for solving ODEs which are linear with analytic non-constant coefficients or non-constant coefficients with regular singular points (recall removable singularities). In our Math 336 class, we will solve first and second order linear equations using PSMs. For nonlinear ODEs Parker and Sochacki of JMU developed a method that uses auxiliary variables and extends power series methods for a wide class of ODEs. You may want to compare to Cauchy-Kowalevski theorem in PDEs.

1 Concepts

Consider the ODE

$$y'' + p(x)y + q(x) = 0.$$

1. **Expansion about a regular point:** If $p(x)$ and $q(x)$ are analytic near x_0, then so is y and we seek a solution of the form $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$. The interval of convergence of the series for y will be the same as the smaller of the interval of convergence of the series for $p(x)$ and $q(x)$ about x_0. Plugging the series solution into the ODE yields recursion formulas for the a_n's allowing us to find the general solution (with 2 arbitrary constants combining 2 linearly independent solutions obtained from the recursions.)

2. **Expansion about a regular singular point:** If $p(x)$ and $q(x)$ are singular near x_0, then the solution near singular point is not easy and needs careful analysis. However, if x_0 is not a 'bad' singular point, called *regular singular point*, then a (*Frobenius*) series solution of the form $y = (x-x_0)^m \sum_{n=0}^{\infty} a_n(x-x_0)^n$ exists, (where m need not be an integer and usually captures the singularity in the solution). Note that $a_0 \neq 0$ in Frobenius series. This series has the same interval of convergence as the smaller of the interval of convergence of the series for $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ about x_0, except perhaps at x_0.

   Here's how Frobenius method works For simplicity, let $x_0 = 0$. Expand $xp(x) = p_0 + p_1 x + p_2 x^2 + \ldots$ and $x^2 q(x) = q_0 + q_1 x + q_2 x^2 + \ldots$. Plugging these and the Frobenius solution $y = x^m \sum_{n=0}^{\infty} a_n x^n$ into the ODE, and collecting the coefficient of a_0, we obtain **the indicial equation**,

   $$m(m-1) + p_0 m + q_0 = 0,$$

   which has two roots m_1 and m_2. (You can immediately start by finding the indicial equation, its roots, and skip to the next step.)

   We have the following cases:

   (a) $m_1 \neq m_2$**, and** $m_1 - m_2$ **is not an integer**. Here we obtain two linearly independent Frobenius series solutions, one corresponding to each m. All we have to do is to substitute m then solve for the rest of the coefficients in the series in terms of a_0.

   (b) $m_1 \neq m_2$**, and** $m_1 - m_2 = N$**, a positive integer, the roots are** m_1 **and** $m_1 + N$**.** Look after you plug in your series, for the coefficients of x^{m+N}. Two things may happen:

      i. If that coefficient looks like $0 * a_N +$ *stuff that also adds up to zero*, then in this case, the smaller root m_1 alone will give two linearly independent solutions, with the arbitrary constants producing these solutions are a_0 and a_N.

ii. If it looks like $0 * a_N +$ *stuff that does not add up to zero*, then in this case, the larger root $m_1 + N$ will give one Frobenius series solution (with arbitrary constant a_0). How do we find the second linearly independent solution? It will be of this form, called a *logarithmic* solution,

$$x^{m_1} * (b_0 + b_1 x + b_2 x^2 + \dots) - b_N * \ln x * (\textit{Frobenius series obtained from the larger root}), x > 0,$$

where the $b's$ are unknowns, which can be found once we plug this new ansatz into the ODE, and noting that $y_1 =$ *Frobenius series obtained from the larger root* does solve the ODE, so it's not a bad substitution.

(c) $m_1 = m_2$. Then only one Frobenius series solution can be found. You can try find another linearly independent solution like the one in case (ii) above, using $N = 0$. That is of the form $x^m * (b_0 + b_1 x + b_2 x^2 + \dots) - b_0 * \ln x * y_1$.

2 Reading assignment

Read chapter 4 from the book.

3 Problem set (due Thursday November 12 2015)

1. (Real analytic functions) What is the difference between a real smooth function and a real analytic function on an interval (a, b)? What does it mean for a function $f(x)$ to be analytic at a point x_0?

2. (Expansion about a regular point) Problem 2 section 4.3. Check the radii of convergence of your series solutions. Could you have known the radii of convergence just from examining ODE?

3. Problem 4 section 4.3.

4. (Chebyshev's equation and polynomials) Problem 7 section 4.3. What are Chebyshev's equation and polynomials useful for?

5. (Hermite's equation and polynomials) Problem 8 section 4.3. What are Hermite's equation and polynomials useful for?

6. (Singular points of ODEs) Problem 1 section 4.4.

7. (Singular points of ODEs) Problem 2 section 4.4.

8. (Frobenius method, indicial equation) Problem 3 part a section 4.4.

9. (Regular singular point , Frobenius method) Problem 4 part c section 4.4.

10. (Bessel's equation, Frobenius method) Problems 6 and 7 section 4.5. What is Bessel's equation useful for?

4 Graphing and computing

1. Write down the Taylor series expansion of the function $f(x) = \ln(1 + x)$ about the point 0. What is the radius of convergence of this power series? Plot $\ln(1 + x)$ and its first four Taylor polynomials on the same graph (label your plot- if using Matlab, use the function LEGEND).

2. Write a program that solves linear second order ODEs with no singular points using PSM.