Math 336 Ordinary Differential Equations Written Assignment 5

Systems of Differential Equations (Linear, Nonlinear)

1 Concepts

1. Transforming higher order equations to first order systems (linear or nonlinear).

1.1 First order linear systems

1. Expressing linear systems in matrix and vector notation $\underline{w}' = A\underline{w} + \underline{f}$ and vice versa.
2. The eigenvalue method to solve linear systems: If $\underline{w} = \underline{v}e^{\lambda t}$ solves the homogeneous linear system $\underline{w}' = A\underline{w}$, then λ must be an eigenvalue of A with corresponding eigenvalue λ. So the method tells us to find the eigenvalues of A using $\det(A - \lambda I) = 0$, then find the corresponding eigenvectors.

   These cases may happen:

   (a) distinct eigenvalues: then the solution is the linear combination of $\underline{v_i}e^{\lambda_i t}$.
   - In the case of complex eigenvalues, since the coefficient matrix is real, they come in conjugate pairs $\lambda_{1,2} = a \pm bi$. To find the two real solutions associated with this pair, use only one complex eigenvalue and its corresponding eigenvector, $\underline{v}e^{at}e^{ibt} = \underline{v}e^{at}(\cos(bt) + i\sin(bt))$ then multiply, separate the real and imaginary parts of the resulting vector solution, these will be the desired real solutions.

   (b) repeated real eigenvalues:

      i. producing enough eigenvectors: the solution is the linear combination of $\underline{v_i}e^{\lambda t}$.

      ii. not producing enough eigenvectors: Not required in Math 336, but the phase portrait is required (very improper node).

   (c) repeated complex eigenvalues: not required in Math 336. Note that the matrix has to be at least 4×4 for this to happen.
3. Plotting solutions:

   (a) Plot each coordinate of $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ as a function of t. Note that periodic solutions produce closed paths in the phase portrait.

   (b) Phase portraits and stability: eigenvalues real (proper nodes and very improper nodes (λ's equal), improper nodes, saddles); eigenvalues complex (centers or spirals).
4. Nonhomogeneous system? Need to find a particular solution $\underline{w_p}$ and add it to the complimentary solution.
5. Numerical methods: Euler, Improved Euler, Runge Kutta.

1.2 First order nonlinear systems

Autonomous system $\left\{\begin{array}{rcl} x'(t) & = & F(x,y), \\ y'(t) & = & G(x,y). \end{array}\right.$

1. Critical points: RHS= $\underline{0}$.
2. Isolated critical points and equilibrium solutions (note that $\underline{0}$ is the only critical point for a homogeneous linear system $\left\{\begin{array}{rcrcl} x'(t) & = & a_1 x & + & b_1 y, \\ y'(t) & = & a_2 x & + & b_2 y, \end{array}\right.$ whenever $\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$).
3. Linearizing around a critical point (x^*, y^*), $x = x^* + u$ and $y = y^* + v$, we get
$$\left\{\begin{array}{rcrcrcl} u'(t) & = & a_1 u & + & b_1 v & + & f(u,v), \\ v'(t) & = & a_2 u & + & b_2 v & + & g(u,v). \end{array}\right.$$
4. Simple critical point at $(0,0)$: isolated and remainder goes to zero, that is, $\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}} = 0$ and $\lim_{(u,v)\to(0,0)} \frac{g(u,v)}{\sqrt{u^2+v^2}} = 0$. This condition is sufficient to guarantee that the linearized system is a good approximation of the original nonlinear system:
   - Find the type and stability of the critical point of the linearized system.
   - This will be the same as for the nonlinear system, except possibly in two *sensitive* cases:
     (a) Eigenvalues real and equal: linearized system node, nonlinear system either node or spiral point (stability determined by the sign of $\lambda_1 = \lambda_2$).
     (b) Eigenvalue purely imaginary: linearized system center, nonlinear system either center or spiral point (could be stable, asymptotically stable, or unstable).

     In these sensitive cases, determining the analytical *implicit* solution by attempting to solve $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$ maybe helpful, even though it defeats the purpose in a way (see problem (2) below).
5. Using the Jacobian $J(x,y) = \begin{pmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{pmatrix}$ to linearize about critical points.
6. Big picture phase portrait.
7. Limit cycles?
8. Applications:
   (a) Ecological models: Interaction of logistic population- competition, cooperation, predation- depending on the signs of c_1 and c_2 in
$$\begin{cases} x'(t) = a_1 x - b_1 x^2 - c_1 xy \\ y'(t) = a_2 y - b_2 y^2 - c_2 xy. \end{cases}$$
   (b) Nonlinear pendulum (undamped: $\theta'' + \omega^2 \sin(\theta) = 0$, damped: $\theta'' + c\theta' + \omega^2 \sin(\theta) = 0$): infinitely many critical points.
9. Separatrix.

2 Reading assignment

Read chapters 10 and 11 from the book.

3 Problem set (due Thursday December 10 2015)

1. The matrix $A = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}$ has an eigenvalue $\lambda_1 = 2i$ and a corresponding eigenvector $\underline{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$.

   (a) Find the real solution of the system $\underline{w}' = A\underline{w}$.

   (b) Each trajectory is an ellipse. Write down the parametric equations of these ellipses: $x(t) = ...$ and $y(t) = ...$

   (c) Along which directions are the major and minor axis of these ellipses are? (Does the eigenvector tell you any information about this?)

   (d) Use a graphing utility to plot the coordinates $x(t)$ and $y(t)$ as a function of t, when the initial conditions are $x(0) = -4$ and $y(0) = -1$.

   (e) Plot the phase portrait.

2. Analyze the type and stability of the critical points of the following predator-prey ecological system, then plot a phase portrait showing all interactions and the separatrices if any.

$$\begin{cases} x'(t) = 7x - x^2 - 2xy \\ y'(t) = y - y^2 + 2xy. \end{cases}$$

3. Analyze the type and stability of the critical points of the following predator-prey ecological system, then plot a phase portrait showing all interactions and the separatrices if any.

$$\begin{cases} x'(t) = 200x - 4xy \\ y'(t) = -150y + 2xy. \end{cases}$$

   Find the analytical solution $f(x, y) = C$ to show that one of the critical points for the nonlinear system above is in fact a center, since it is a sensitive case. You may need a graphing utility or MATLAB for the contour plot of $f(x, y)$.

4. Consider the damped nonlinear pendulum

$$\theta'' + 0.1\theta' + \sin(\theta) = 0.$$

   Transform into a first order system, find the critical points, linearize to study the types and the stability of the critical points, then plot a phase portrait, showing the separatrices.

4 Graphing and computing

1. For number (3) above, use MATLAB to find the contour plot for the analytical solution $f(x, y) = C$ of the system.

2. Use a phase portrait graphing utility to plot the phase portraits of the systems in problems 1, 2, 3 and 4 above, hence confirming your predictions.