Math 336 Ordinary Differential Equations Written Assignment 6

ODE Theory. Existence and Uniqueness. Picard's Iteration

1 Concepts

1.1 First order ODEs existence and uniqueness

The following first order ODE could be linear or nonlinear, homogeneous or no homogeneous,

$$\frac{dy}{dx} = F(x, y).$$

1. **Picard's theorem** If $F(x, y)$ is continuously differentiable on a domain $(a, b)\times(c, d)$, bounded, and satisfies a Lipschitz condition in y, that is,

   $$|F(x, s) - F(x, t)| \leq C|s - t|.$$

   (Note that if the partial derivative $\frac{\partial F}{\partial y}$ is bounded then the above Lipschitz condition is automatically satisfied.) Let $x_0 \in (a, b)$ and $y_0 \in (c, d)$, then there is an $h > 0$ such that $(x_0 - h, x_0 + h) \subset (a, b)$, and a continuously differentiable function y on $(x_0 - h, x_0 + h)$ taking values in (c, d) that solves the initial value problem $\frac{dy}{dx} = F(x, y)$ with initial conditions $y(x_0) = y_0$. (Note that this guarantees existence only in a small neighborhood of x_0, not in all of (a, b).) The solution is unique in the sense that if $\tilde{y}$ is another continuously differentiable function on some interval $(x_0 - \tilde{h}, x_0 + \tilde{h})$ that solves the initial value problem, then $y = \tilde{y}$ on the common interval.

   Note that if the Lipschitz condition is dropped on $F(x, y)$, then we lose uniqueness but we still have existence.

2. **Picard's iteration** The main idea is that $\frac{dy}{dx} = F(x, y)$ together with $y(x_0) = y_0$ are equivalent, using the fundamental theorem of calculus, to the single integral equation,

   $$y(x) = y_0 + \int_{x_0}^{x} F(s, y(s))ds.$$

   Now we construct a sequence of functions $\{y_1(x), y_2(x), \dots\}$ that will converge uniformly on an interval of the form $(x_0 - h, x_0 + h) \subset (a, b)$,

   $$y_{j+1}(x) = y_0 + \int_{x_0}^{x} F(s, y_j(s))ds,$$

   where $y_0(s) = y_0$. The limit $\lim_{j\to\infty} y_j(x) = y(x)$ of the sequence is continuously differentiable and it's the unique solution of the IVP on that interval.

3. **Proof of the above is required** (section 3.3.4 in the book).

1.2 Higher order linear ODEs: existence and uniqueness

1. **Linear independence** of functions $f_1, f_2, \dots, f_n$ on an interval $[a, b]$: when there *do not exist* constants c_i's such that $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ at *each* point of $[a, b]$. So there is no *linear relationship* between the functions, or we can't solve for one linearly in terms of the others.

2. **The Wronskian** of functions $W(f_1, f_2, \dots, f_n)$, and its relationship to their linear independence (non-vanishing of the Wronskian). If y_n's solve an nth order linear ODE, then their Wronksian is either nowhere vanishing (in this case they are linearly independent) or vanishes identically (in this case they are linearly dependent).

3. **The general solution** If we have n linearly independent solutions of an nth order linear homogeneous ODE, then their linear combination is also a solution. Moreover (and we have been using this idea for all the higher order linear ODEs that we solved this semester) any other solution is a linear combination of those. That is, the general solution will be given by the linear combination of these n linearly independent solutions.

4. **Existence and Uniqueness** This follows from Picard's theorem for first order ODEs. If the coefficients of nth order linear ODE (homogeneous or non homogeneous) are continuous functions on $[a, b]$, and $x_0 \in [a, b]$, then there *exists a unique* solution of the ODE with $y(x_0) = y_0$, $y'(x_0) = y_1$, $\ldots$, $y^{(n-1)}(x_0) = y_{n-1}$.

1.3 Extra and not required in Math 336. Second order ODEs- Roots of solutions, oscillations, Sturm separation and comparison theorems

- *Zeros of linearly independent solutions*: If y_1 and y_2 are two linearly solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

  then the zeros of y_1 and y_2 are distinct and occur alternately.

- *Non-oscillatory solution*: If $q(x) < 0$ for all x and y is a nontrivial solution of

$$y'' + q(x)y = 0,$$

  then y has at most one zero.

- *Oscillatory solution*: If $q(x) > 0$ for all $x > 0$, $\int_1^\infty q(x)dx = \infty$ and y is a nontrivial solution of

$$y'' + q(x)y = 0,$$

  then y has infinitely many zeros on the positive x-axis.

2 Reading assignment

Read chapter 3 from the book.

3 Problem set (Not due)

1. Performing Picard's iteration and checking that it converges to the solution of an ODE if you know the solution. (Similar to examples in section 3.3.3 in the book.)
2. Estimating Picard's iterates, or the proof of Picard's theorem. (Section 3.3.4 in the book.)

4 Graphing and computing

1. A MAPLE or MATHEMATICA routine to find Picard's iterates.