NYU MathUA-233 Theory of Probability (Summer 2017 Session II)

(Final Exam) Thurs August 10, 2017

Name:

1. (2 points) Compute the probability that a hand of 13 cards contains:
   (a) The ace and king of at least one suit.
   (b) All four of at least one of the 13 denominations.

2. (2 points) A bin contains 3 types of disposable flashlights. The probability that a type 1 flashlight will give more than a hundred hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 flashlights being 0.4 and 0.3 respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

   (a) What is the probability that a randomly chosen flashlight will give more than a hundred hours of use?

   (b) Given that a flashlight lasted more than a hundred hours, what are the conditional probabilities that it was type j flashlight, $j = 1, 2, 3$?

3. (2 points) Suppose that X and Y are independent, continuous random variables having probability density functions f_X and f_Y respectively.

   (a) Find the CDF of $Z = X + Y$ as an integral formula.

   (b) Find the PDF of $Z = X + Y$ as an integral formula.

4. (3 points) Let X and Y be independent $Poiss(\lambda)$ random variables. Recall that the moment generating function of X is $M(t) = e^{\lambda(e^t-1)}$.

   (a) Find the moment generating function of $X + 2Y$ (simplify).

   (b) Is $X + 2Y$ also Poisson? Show it is or isn't, whichever is true.

   (c) Let $g(t) = \ln M(t)$. Expanding $g(t)$ as a Taylor series, $g(t) = \sum_{j=1}^{\infty} \frac{c_j}{j!} t^j$, the coefficient c_j is called the jth cumulant of X. Find a simplified formula for c_j in terms of λ, for all $j \geq 1$.

5. (2 points) Two gamblers A and B are successively betting a dollar on each round of a gambling game, in which either A or B wins each round. Assume each of A and B starts with a finite amount of money, the game ends when one of the gamblers goes bankrupt and that the probability that A wins a round is p.

   (a) What is the probability that A wins the entire game?

   (b) Show that A's sequence of fortunes is a Markov chain and set up its transition matrix.

6. (3 points) An astronomer is interested in measuring the distance, in light years, from his observatory to a distant star. The astronomer plans to make a series of measurements and then use the average value of these measurements as his estimated value of the actual distance. If the astronomer believes that the values of the measurements are independent and identically distributed random variables having a common mean d (the actual distance) and a common variance 4 light years, how many measurements need he make to be reasonably sure that his estimated distance is accurate to within ± 0.5 light year?

7. (2 points)

   (a) Prove Markov's inequality $P(X \geq a) \leq \frac{E(X)}{a}$ where X is a nonnegative random variable and $a > 0$.

   (b) Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50. What can be said about the probability that this week's production will exceed 75?

8. (2 points) In a large group of n people, find the *approximate* probability that at least 3 people end up with the same birthday. (Hint: you can define a Poisson random variable.)

9. (2 points) Derive Stirling's formula

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$

for large n using the normal approximation of Poisson distribution (justify all your steps).

10. (2 points) A chicken lays n eggs. Each egg independently does or does not hatch, with probability p of hatching. For each egg that hatches, the chick does or does not survive, with probability s for survival. Let $N \sim Bin(n, p)$ be the number of eggs which hatch, X be the number of chicks who survive, and Y the number of chicks which hatch but do not survive (so $X + Y = N$).

    (a) Find the marginal PMF of X.

    (b) Find the joint PMF of X and Y. Are they independent?

11. (2 points) For a Poisson process $N(t)$, let T_1 denote the time the first event occurs. Further, for $n > 1$, let T_n denote the time elapsed between the $(n-1)$th and the nth event. The sequence $\{T_n\}_{n=1}^{\infty}$ is called the *sequence of inter-arrival times*. Prove that $T_1, T_2, \ldots$ are independent exponential random variables, each with mean $1/\lambda$.

12. (2 points) Let $X \sim Geom(p)$. Find the moment generating function of X.

13. (2 points) A stick is broken into two pieces at a uniformly chosen random break point. Find the CDF and the average length of the longer piece.

14. (2 points) A random point (X, Y, Z) is chosen uniformly in the three dimensional ball $B = \{(x, y, x) : x^2 + y^2 + z^2 \leq 1\}$.

    (a) Find the joint PDF of X, Y, Z.

    (b) Find the joint PDF of X, Y.

15. (2 points) List the most important properties that you learned about each of the following distributions and how they relate to other distributions that you know.

    (a) $U \sim Unif(0, 1)$.

    (b) $X \sim N(\mu, \sigma^2)$.

    (c) $X \sim Poiss(\lambda)$.

    (d) $X \sim Exp(\lambda)$.