NYU MathUA-233 Theory of Probability (Summer 2017 Session II)

(Week 5) Mon July 31- Thurs August 3, 2017

1 Jointly Distributed Random Variables (Chapters 6 and 7)

1. **Universality of the uniform distribution**: Let $X \sim F$ be a continuous random variable with CDF $F(x)$, then the random variable $Y = F(X)$ is uniformly distributed: $Y \sim Unif(0, 1)$. That is equivalent to saying that the CDF of $F^{-1}(U)$ where $U \sim Unif(0, 1)$ is the function F. Therefore, to generate a continuous random variable X with a desired distribution F, all you have to do is generate a random variable $U \sim Unif(0, 1)$, then set $X = F^{-1}(U)$ (so generate a random number u, then set $u = F(x)$, then solve for x).

2.
    - Joint CDF and marginal CDFs (for both discrete and continuous random variables).
    - Joint PDF and marginal PDFs (for continuous random variables).
    - Joint PMF and marginal PMFs (for discrete random variables).

3. **Independent Random Variables** X and Y are independent if knowing the value of one does not change the distribution of the other.

    (a) X and Y are independent *iff* $P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$ *iff* $F_{X,Y}(a, b) = F_X(a)F_Y(b)$ for all $a, b \in \mathbb{R}$. (This is valid for both discrete and continuous random variables.)

    (b) For discrete random variables: Independent *iff* $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all $x, y \in \mathbb{R}$.

    (c) For jointly continuous random variables: Independent *iff* $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbb{R}$.

    (d) The independence of more than two random variables $X_1, X_2, \ldots, X_n$ can be established sequentially: Show X_2 is independent of X_1, then X_3 is independent of X_1, X_2, ..., then X_n is independent of $X_1, X_2, \ldots, X_{n-1}$.

    (e) If X and Y are independent, then for any functions h and g, $E(h(X)g(Y)) = E(h(X))E(g(Y))$. In particular $E(XY) = E(X)E(Y)$.

4. **Sum of independent** (and sometimes identically distributed random variables (i.i.d.s)): We need to calculate the distribution of $X_1 + X_2 + \cdots + X_n$ given that we know the distribution of each of the X_i's, $i = 1, 2, \ldots, n$.

    (a) Convolution: $f * g(a) = \int_{-\infty}^{\infty} f(a - y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(a - y)dy$.

    (b) Sum of uniformly distributed i.i.d.s.

    (c) Sum of normally distributed independent random variables.

5. **Covariance**

$$Cov(X, Y) = E((X - \mu_X)(Y - \mu_Y)) = E(XY) - E(X)E(Y) = \mu_{XY} - \mu_X\mu_Y$$

- If X and Y are independent, then $Cov(X,Y) = 0$ since in that case $E(XY) = E(X)E(Y)$. The inverse of this statement need not be true.
- Covariance is bilinear.
- Covariance has same dimension as that of XY.

6. **Correlation** is normalized covariance (so it is a dimensionless quantity).

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

7. PDF of *functions of* jointly distributed random variables (Jacobian of the transformation).

8. **Expectation** of *functions* of random variables $E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$.

2 Reading assignment

Chapter 6 sections: 6.1, 6.3, 6.3, 6.7.
Chapter 7 sections: 7.1, 7.2, 7.3, 7.4, 7.7, 7.8, 7.9.

3 Homework Assignment

(Due Monday August 7)

1. Page 272 (Problems): 6.22, 6.38, 6.53.

2. Page 275 (Theoretical problems): 6.2, 6.10.

3. Page 354 (Problems): 7.26.

4. Page 359 (Theoretical problems): 7.11, 7.24.

5. A stick is broken into two pieces, at a uniformly random chosen break point. Find the CDF and the average length of the longer piece. (*Ans*: $L \sim Unif(1/2, 1)$ and $E(L) = 3/4$.)

6. Two students A and B are working independently on homework. Student A takes $Y_1 \sim Exp(\lambda_1)$ hours to finish her homework, and student B takes $Y_2 \sim Exp(\lambda_2)$ hours to finish the homework.

   (a) Find the CDF and PDF of $\frac{Y_2}{Y_1}$, the ratio of their problem solving times.

   (b) Find the probability that A finishes her homework before B does.

4 Test 5 on Thursday August 3, 2017

Name:

1. (1 point) *Universality of the Uniform*: Let $X \sim F$ be a continuous random variable with distribution function F. Prove that for $Y = F(X)$, $E(Y) = \frac{1}{2}$.

2. (2 points) *Distribution of transformation- Jacobian*: Let $U \sim Unif(0, 2\pi)$, and $Z \sim Exp(1)$ be independent of U.

   (a) Find the joint PDF of X and Y defined by $X = \sqrt{2Z}\cos(U)$ and $Y = \sqrt{2Z}\sin(U)$.

   (b) Deduce that X and Y are independent standard normal random variables.

3. (2 points) *Expectation*: Let $X, Y \sim Exp(\lambda)$ be i.i.d. exponentially distributed random variables. Find $E(|X - Y|)$.

4. (1 point) *Multinomial distribution*: A fair die is rolled 9 times. Let $X_1, X_2, X_3, X_4, X_5, X_6$ be respectively the random variables counting the number of times each of $1, 2, 3, 4, 5, 6$ appear. Find the joint PMF of $X_1, X_2, X_3, X_4, X_5, X_6$. Deduce $P(A)$ where A is the event that 2 appears 3 times, 4 appears 4 times and 6 appears 2 times.

5. (2 points) *i.i.d. random variables*: Let U_1, U_2, U_3, U_4 be i.i.d. $Unif(0, 1)$ random variables, and let $L = \min(U_1, U_2, U_3, U_4)$ and $M = \max(U_1, U_2, U_3, U_4)$.

   (a) Find the marginal CDF and the marginal PDF of M.

   (b) Find the joint CDF and the joint PDF of M, L.

6. (1 point) *Covariance and correlation*: Two fair six-sided dice are rolled, with outcomes X and Y respectively for die 1 and die 2.

   (a) Compute the covariance of $X + Y$ and $X - Y$.

   (b) Are $X + Y$ and $X - Y$ independent? Show that they are or that they aren't.

7. (1 point) *Normally distributed random variables*: Let $X_i \sim N(1, 2), i = 1, 2, \ldots, n$ be i.i.d normally distributed random variables. Find $E(4X_1 - X_2 + X_3)$ and $Var(2(X_1 + X_2 + \cdots + X_n))$.