① $\displaystyle\int_{-1}^{1} \frac{e^{\tan^{-1}y}}{1+y^2}\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^u\,du$

$$\left(\begin{array}{l} u=\tan^{-1}y \\ du=\dfrac{1}{1+y^2}\,dy \\ \int_{-1}^{1} \to \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \end{array}\right)$$

$= e^u \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \boxed{e^{\frac{\pi}{4}} - \dfrac{1}{e^{\frac{\pi}{4}}}}$

② $\displaystyle\int x\sin^2 x\,dx$

$= \displaystyle\int x\cdot\frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\displaystyle\int x - x\cos 2x\,dx$

$= \frac{1}{2}\left[\displaystyle\int x\,dx - \int \underset{D}{x}\,\underset{I}{\cos 2x}\,dx\right]$

$= \frac{1}{2}\left[\frac{1}{2}x^2 - \left(\frac{1}{2}\sin 2x\cdot x - \displaystyle\int \frac{1}{2}\sin 2x\,dx\right)\right]$

by Integration by Parts.

$= \boxed{\frac{1}{4}\left(x^2 - x\sin 2x + \left(-\frac{1}{2}\cos 2x\right)\right) + C}$

③ $\displaystyle\int (1+\sqrt{x})^8\,dx = \int (1+\sqrt{x})^8\,\frac{2\sqrt{x}}{2\sqrt{x}}\,dx$

$$\left(\begin{array}{l} u=1+\sqrt{x} \\ du=\dfrac{1}{2\sqrt{x}}\,dx \\ \sqrt{x}=u-1 \end{array}\right)$$

$= \displaystyle\int u^8\cdot 2(u-1)\,du$

$= 2\displaystyle\int u^9 - u^8\,du$

$= 2\left(\frac{1}{10}u^{10} - \frac{1}{9}u^9\right) + C$

$= \boxed{\frac{2}{10}(1+\sqrt{x})^{10} - \frac{2}{9}(1+\sqrt{x})^9 + C}$

④ $\displaystyle\int_0^{\frac{1}{2}} \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^{\frac{1}{2}} \sqrt{\frac{1+x}{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

$= \displaystyle\int_0^{\frac{1}{2}} \frac{1+x}{\sqrt{1-x^2}}\,dx$

$$\left(\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\,d\theta \\ \int_0^{\frac{1}{2}} \to \int_{\sin^{-1}0}^{\sin^{-1}\frac{1}{2}} = \int_0^{\frac{\pi}{6}} \end{array}\right)$$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta$

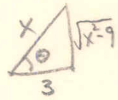
$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{1+\sin\theta}{|\cos\theta|}\cdot\cos\theta\,d\theta$

$= \displaystyle\int_0^{\frac{\pi}{6}} 1+\sin\theta\,d\theta$, since $|\cos\theta| = \cos\theta$ as $0 \le \theta \le \frac{\pi}{6}$.

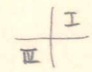
$= \theta - \cos\theta\Big|_0^{\frac{\pi}{6}} = \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) - (0-1)$

$= \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1}$.

⑤ $\displaystyle\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\,dx$



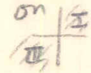
$$\left(\begin{array}{l} x=3\sec\theta \\ dx=3\sec\theta\tan\theta\,d\theta \end{array}\right.$$



$= \displaystyle\int \frac{1}{(9(\sec^2\theta-1))^{\frac{3}{2}}}\cdot 3\sec\theta\tan\theta\,d\theta$

$= \displaystyle\int \frac{3\sec\theta\tan\theta}{\sqrt{9\tan^2\theta}^{\,3}}\,d\theta$

$= \displaystyle\int \frac{3\sec\theta\tan\theta}{(3\tan\theta)^3}\,d\theta$, since $\tan\theta \ge 0$ on



$= \displaystyle\int \frac{\sec\theta}{9\tan^2\theta}\,d\theta$

$= \frac{1}{9}\displaystyle\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{u^2}\,du = \frac{1}{9}\left(-\frac{1}{u}\right) + C = -\frac{1}{9\sin\theta} + C$

$u=\sin\theta$
$du=\cos\theta\,d\theta$

$= \boxed{-\frac{1}{9}\cdot\frac{x}{\sqrt{x^2-9}} + C}$

⑥ $\int x^5 e^{-x^3}\,dx \quad \begin{pmatrix} u = x^3 \\ du = 3x^2\,dx \\ \frac{1}{3}du = x^2\,dx \end{pmatrix}$

$= \int x^3 e^{-x^3} \cdot x^2\,dx$

$= \frac{1}{3}\int \underset{D}{u}\, \underset{I}{e^{-u}}\,du$

$= \frac{1}{3}\left[-e^{-u}u - \int -e^{-u}du\right]$

$= -\frac{1}{3}ue^{-u} + \frac{1}{3}(-e^{-u}) + C$

$= \boxed{-\frac{1}{3}x^3 \cdot e^{-x^3} - \frac{1}{3}e^{-x^3} + C}$

⑦ $\int \frac{1}{x\sqrt{4x+1}}\,dx \quad \begin{pmatrix} u = \sqrt{4x+1} \ : \ \frac{u^2-1}{4} = x \\ du = \frac{\cancel{4}\,2}{\cancel{2}\sqrt{4x+1}}\,dx \\ \frac{1}{2}du = \frac{1}{\sqrt{4x+1}}\,dx \end{pmatrix}$

$= \int \frac{\cancel{4}\,2}{u^2-1} \cdot \frac{1}{\cancel{2}}du$

$= \int \frac{2}{u^2-1}\,du = \int \frac{1}{u-1} - \frac{1}{u+1}\,du$

$= \ln|u-1| - \ln|u+1| + C$

$= \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$

$= \boxed{\ln\left|\frac{(\sqrt{4x+1}-1)^2}{4x}\right| + C}$

⑧ $\int x\sqrt[3]{x+c}\,dx \quad \begin{pmatrix} u = x+c \\ du = dx \\ x = u-c \end{pmatrix}$

$= \int (u-c)\sqrt[3]{u}\,du$ $\quad$ c: some fixed real #

$= \int (u-c)u^{\frac{1}{3}}\,du$

$= \int u^{\frac{4}{3}} - cu^{\frac{1}{3}}\,du$

$= \frac{3}{7}u^{\frac{7}{3}} - c\cdot\frac{3}{4}u^{\frac{4}{3}} + C = \boxed{\frac{3}{7}(x+c)^{\frac{7}{3}} - \frac{3c}{4}(x+c)^{\frac{4}{3}} + D}$, where D is some constant.
(since we're using c for something else in this problem)

⑨ $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$ $\quad$ $u = \sqrt{x}$, so $u^2 = x$; $du = \frac{1}{2\sqrt{x}}\,dx$; $2du = \frac{1}{\sqrt{x}}\,dx$

$= \int \frac{x}{\sqrt{x}}e^{\sqrt{x}}\,dx$

$= \int u^2 \cdot e^u\, 2du$

$= 2\int \underset{D}{u^2}\,\underset{I}{e^u}\,du$

$= 2\left[e^u \cdot u^2 - \int e^u\, 2u\,du\right]$

$= 2e^u u^2 - 2\cdot 2\int \underset{I}{e^u}\,\underset{D}{u}\,du$

$= 2e^u u^2 - 4\left[e^u u - \int e^u\,du\right]$

$= 2e^u u^2 - 4e^u u + 4e^u + C$

$= 2e^u(u^2 - 2u + 2) + C$

$= \boxed{2e^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C}$

⑩ $\int \frac{e^{2x}}{1+e^x}\,dx$ $\quad$ $u = e^x$; $du = e^x dx$

$= \int \frac{e^x \cdot e^x}{1+e^x}\,dx$

$= \int \frac{u}{1+u}\,du$

$= \int 1 - \frac{1}{1+u}\,du$

$= u - \ln|1+u| + C$

$= \boxed{e^x - \ln(1+e^x) + C}$