1. Show that $f(x) = e^x$ is represented by its Maclaurin series for all $x \in (-\infty, \infty)$.

Step 1. Find the Maclaurin series generated by $f(x) = e^x$ and find its interval of convergence.

Since $f^{(n)}(x) = e^x$ for all non-negative integer n, we have that $f^{(n)}(0) = e^0 = 1$. Therefore, the Maclaurin series generated by e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{1}$$

The interval of convergence of this power series is $(-\infty, \infty)$, since

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1,$$

for all $x \in (-\infty, \infty)$.

Step 2. Use the Taylor's formula to get a formula for e^x.

By the Taylor's Formula, we have that, for each $x \in (-\infty, \infty)$,

$$\begin{aligned} e^x &= P_n(x) + R_n(x), \\ &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}) + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}, \\ &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}) + \frac{e^c}{(n+1)!} x^{n+1}, \end{aligned}$$

for some real number c.

Step 3. Notice that by Step 2, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

for all $x \in (-\infty, \infty)$ if we show that $\lim_{n\to\infty} \frac{e^c}{(n+1)!} x^{n+1} = 0$.

Step 4. Conclude $\lim_{n\to\infty} \frac{e^c}{(n+1)!} x^{n+1} = 0$ by showing that $\lim_{n\to\infty} |\frac{e^c}{(n+1)!} x^{n+1}| = 0$.

Since $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, we have that

$$\lim_{n\to\infty} \left|\frac{e^c}{(n+1)!} x^{n+1}\right| = e^c \lim_{n\to\infty} \frac{|x^{n+1}|}{(n+1)!} = e^c \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^c \cdot 0 = 0,$$

for all $x \in (-\infty, \infty)$.

Hence, $\lim_{n\to\infty} \dfrac{e^c}{(n+1)!}x^{n+1} = 0$ and therefore,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

for all $x \in (-\infty, \infty)$, and we are done.