#1 $\displaystyle\int \frac{1-\tan x}{1+\tan x}\,dx$

$$= \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x-\sin x}{\cos x+\sin x}\,dx$$

$\begin{cases} u = \cos x+\sin x \\ du = (-\sin x+\cos x)\,dx \end{cases}$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\cos x+\sin x| + C}$$

#2 $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}$, where $\boxed{a>0}$ is some fixed #.

$\begin{cases} x = a\sec\theta, \quad 0\le\theta<\frac{\pi}{2} \text{ or } \pi\le\theta<\frac{3\pi}{2} \\ dx = a\sec\theta\tan\theta\,d\theta \end{cases}$

$$= \int \frac{1}{\sqrt{a^2\sec^2\theta-a^2}}\cdot a\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{a^2(\sec^2\theta-1)}}\cdot a\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{a|\tan\theta|}\cdot a\sec\theta\tan\theta\,d\theta$$

since $a>0$, $\sqrt{a^2}=|a|=a$.

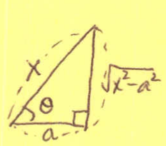
$$= \int \frac{1}{\cancel{a}\cancel{\tan\theta}}\cdot \cancel{a}\sec\theta\cancel{\tan\theta}\,d\theta$$

since $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$, $\tan\theta\ge 0$.

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| + C}.$$



$\sec\theta = \frac{x}{a}$

so $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$.

#3 $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

note

$3-2x-x^2 = 3-(x^2+2x)$
$= 3-(x^2+2x+1)+1$
$= 4-(x+1)^2$

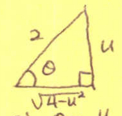
$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$\begin{cases} u = x+1 \\ du = dx \end{cases}$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$\begin{cases} u = 2\sin\theta, \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \\ du = 2\cos\theta\,d\theta \end{cases}$

$$= \int \frac{2\sin\theta-1}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta$$



$\sin\theta = \frac{u}{2}$

$\cos\theta = \frac{\sqrt{4-u^2}}{2}$

$$= \int \frac{2\sin\theta-1}{\sqrt{4(1-\sin^2\theta)}}\cdot 2\cos\theta\,d\theta$$

$$= \int \frac{2\sin\theta-1}{\cancel{2}|\cancel{\cos\theta}|}\cdot \cancel{2\cos\theta}\,d\theta$$

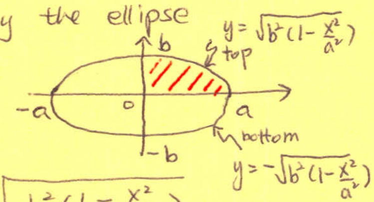
⊛ Since $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $\cos\theta\ge 0$, so $|\cos\theta| = \cos\theta$.

$$= -2\cos\theta - \theta$$

$$= -\cancel{2}\cdot\frac{\sqrt{4-u^2}}{\cancel{2}} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \boxed{-\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C}$$

#4 Show that the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab. $(a, b>0)$



$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y^2 = b^2\left(1-\frac{x^2}{a^2}\right) \rightarrow y = \pm\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$

The area of the shaded region is, therefore, $\int_0^a \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\,dx$

and so the area of the ellipse is

$$4\cdot\int_0^a \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}\,dx$$

$\begin{cases} u = \frac{x}{a} \\ du = \frac{1}{a}\cdot dx \quad \text{so} \quad a\cdot du = dx \\ \int_0^a \rightarrow \int_0^1 \end{cases}$

$$= 4\int_0^1 \sqrt{b^2(1-u^2)}\cdot a\,du$$

$$= 4ab\int_0^1 \sqrt{1-u^2}\,du \qquad (\sqrt{b^2}=|b|=b, \text{ since } b>0)$$

$$= 4ab\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$\begin{cases} u = \sin\theta, \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \text{ so} \\ du = \cos\theta\,d\theta, \quad \cos\theta\ge 0. \end{cases}$

$$= 4ab\int_0^{\frac{\pi}{2}} |\cos\theta|\cdot\cos\theta\,d\theta$$

$\boxed{\theta = \sin^{-1}u \quad \int_0^1 \rightarrow \int_{\sin^{-1}0}^{\sin^{-1}1} = \int_0^{\frac{\pi}{2}}}$

$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{4ab}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2ab\left[\left(\frac{\pi}{2}+\frac{1}{2}\underbrace{\sin\pi}_{0}\right)-(0+0)\right] = \cancel{2}ab\cdot\frac{\pi}{\cancel{2}} = \boxed{\pi ab}.$$