MATH 236 CALCULUS 2
BONUS QUIZ 1

1. Prove the Integral Test.

(Integral Test) Suppose $\{a_n\}$ is a positive decreasing sequence, and let f be a continuous function such that $a_n = f(n)$ for all positive integer n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

2. Prove the Ratio Test.

(Ratio Test) Let $\{a_n\}$ be a positive sequence. Let

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L,$$

for some $L \in \mathbb{R}$.

(1) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(2) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
(3) If $L = 1$, then $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

step 1 We first prove (1). Suppose $L < 1$. Let $r \in \mathbb{R}$ that satisfies $L < r < 1$. Why does such r always exists?

step 2 Write down the definition of $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$. Next, pick $\epsilon > 0$ to be $r - L$.

step 3 Manipulate the inequality you have in step 2 to get $a_{n+1} < ra_n$ for all $n \geqslant N$ for some positive integer N. Then start plugging in $n = N, N+1, N+2, N+3, \cdots$.

step 4 Add the inequalities in step 3 and explain why $\sum_{n=1}^{\infty} a_n$ is convergent.

step 5 Next, we prove (2). Suppose $L > 1$. Again, write down the definition of $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$, and pick $\epsilon > 0$ to be $L - 1$ this time. Use this to show that $a_n < a_{n+1}$ for $n \geqslant N$ for some positive integer N. Finish the proof by concluding that $\lim_{n\to} a_n \neq 0$.

step 6 Lastly, we prove (3). Suppose $L = 1$. Find L for $\sum_{n=1}^{\infty} \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$. Why does this complete the proof?

step 7 Write a complete mathematical proof of the Ratio Test.