**MATH 236 CALCULUS 2**
**CONVERGENCE OF SEQUENCES**

1. Show that if $\{a_n\}$ is a convergent sequence, then for any positive number $\epsilon$ there exists a corresponding integer $N$ such that

$$|a_n - a_m| < \epsilon$$

for all $n > N$ and $m > N$.

**step 0** Suppose $\{a_n\}$ converges to $L$. Write down what this means. In other words, write down the definition of

"A sequence $\{a_n\}$ converges to $L$."

**step 1** Since the definition in step 0 holds for any $\epsilon > 0$, the statement will be true even if you replace $\epsilon$ by $\frac{\epsilon}{2}$. Rewrite the above inequality with $\epsilon$ replaced by $\frac{\epsilon}{2}$ and convince yourself that there exists some positive integer $N$ such that this inequality holds for all $n > N$.

**step 2** Notice that the inequality in step 1 will hold even if you replace the $n$ by another variable $m$ as long as $m > N$. Write down this inequality that holds for all $m > N$.

**step 3** Recall the triangle inequality:

$$|x + y| \leqslant |x| + |y| \text{ for all } x, y \in \Re.$$

Apply the triangle inequality to $a_n - L$ and $L - a_m$ and conclude your argument.

**step 4** Write a complete mathematical proof by putting together the above steps.

2. (Uniqueness of Limits) Suppose a sequence $\{a_n\}$ converges to $L_1$ and $L_2$. Show that $L_1$ and $L_2$ must be equal.

**step 0**
We will assume the opposite of what we would like to prove and then reach a contradiction. That is, assume that $L_1 \neq L_2$, and then reach a false result. This will show that our assumption was wrong proving that $L_1 = L_2$. Such kind of proof technique is called "proof by contradiction."

**step 1** Write down the definition of $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$.

**step 2**
Assume that $L_1 \neq L_2$. Without loss of generality, assume that $L_1 < L_2$. Then, $L_2 - L_1 > 0$. Choose $\epsilon = \dfrac{L_2 - L_1}{2}$ in step 1.

**step 3** Apply the triangle inequality to $L_2 - a_n$ and $a_n - L_1$ to reach $L_2 - L_1 < L_2 - L_1$ which is a contradiction. This implies that our assumption $L_1 \neq L_2$ is false. Therefore, $L_1 = L_2$, and this completes the proof.

**step 4** Write a complete mathematical proof by putting together the above steps.