MATH 236 CALCULUS 2
LENGTH OF A PLANE CURVE

1. Show that the length of the smooth curve given by the parametric equation

$$x = f(t), y = g(t), a \leqslant t \leqslant b$$

is

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2}dt \text{ or } \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt.$$

step 0 Suppose the following graph represents the graph of the given curve. (See figure 6.22 on p409 in the textbook.)

The plan is to first approximate the length of the curve by adding the length of the line segments.

Divide the curve into n-pieces, and let $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ be the t-values that correspond to each point that divides the curve.

step 1 Let us first consider the first line segment. (See figure 6.23 on p409 in the textbook.)

Discuss with your group why the length of this line segment is $\sqrt{f'(t_1^*)^2 + g'(t_1^{**})^2}\Delta t_1$, where t_1^* and t_1^{**} are some points in (t_0, t_1), and $\Delta t_1 = t_1 - t_0$. (Hint: Since f and g are continuous on $[t_0, t_1]$ and differentiable on (t_0, t_1), by the Mean Value Theorem, there exist some $t_1^*, t_1^{**} \in (t_0, t_1)$ such that

$$\frac{f(t_1) - f(t_0)}{t_1 - t_0} = f'(t_1^*) \text{ and } \frac{g(t_1) - g(t_0)}{t_1 - t_0} = g'(t_1^{**}).)$$

In general, the length of the i-th line segment, for all $1 \leqslant i \leqslant n$, is $\sqrt{f'(t_i^*)^2 + g'(t_i^{**})^2}\Delta t_i$, where t_i^* and t_i^{**} are some points in (t_{i-1}, t_i), and $\Delta t_i = t_i - t_{i-1}$.

step 2 By step 1, we can now approximate the length of the curve by $\sum_{i=1}^{n}\sqrt{f'(t_i^*)^2+g'(t_i^{**})^2}\Delta t_i$. It is clear that as n gets larger (as the curve is divided into smaller pieces), the above summation gives a better approximation to the length of the curve. Therefore, we find the length of the curve by

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{f'(t_i^*)^2+g'(t_i^{**})^2}\Delta t_i=\int_a^b\sqrt{f'(t)^2+g'(t)^2}dt.$$

Since $f'(t)=\dfrac{dx}{dt}$ and $g'(t)=\dfrac{dy}{dt}$, this integral can also be written as

$$\int_a^b\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}dt.$$

2. Show that the length of the curve $y=f(x)$ where $a\leqslant x\leqslant b$ is

$$\int_a^b\sqrt{1+f'(x)^2}dx$$

or

$$\int_a^b\sqrt{1+(\frac{dy}{dx})^2}dx.$$

Hint: Consider $y=f(x), a\leqslant x\leqslant b$ as the parametric equation $x=t, y=f(t), a\leqslant t\leqslant b$.

3. Define the curve length function $s(x)=\int_a^x\sqrt{1+f'(u)^2}du$, where a is some fixed real number. Show that

$$\frac{ds}{dx}=\sqrt{1+f'(x)^2}.$$

Hint: Use the Fundamental Theorem of Calculus Part 1.

Formulas

Length of a smooth curve

- $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$, if $x = f(t), y = g(t), a \leqslant t \leqslant b$.

- $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$, if $y = f(x), a \leqslant x \leqslant b$.

- $\int_c^d \sqrt{(\frac{dx}{dy})^2 + 1} dy$, if $x = g(y), c \leqslant y \leqslant d$.

The derivative of the curve length function s

- $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$

- $\frac{ds}{dy} = \sqrt{1 + (\frac{dx}{dy})^2}$

- $\frac{ds}{dt} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$