A New Approach to the Analysis of Axisymmetric Problems

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A New Approach to the Analysis of Axisymmetric Problems

MINAH OH†
JAMES MADISON UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS,
HARRISONBURG, VA, USA

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This paper builds a new framework to analyze axisymmetric problems through differential forms and exterior calculus. We construct a new weighted $L^2$ de Rham complex that arises when analyzing axisymmetric problems. By constructing its dual complex, we obtain a Hodge decomposition and a Poincaré inequality in weighted spaces, and the well-posedness of the weighted mixed Hodge Laplacian problem. The stability and convergence of the discrete weighted mixed Hodge Laplacian problem are also achieved by using bounded cochain projections.

Keywords: Finite element exterior calculus, axisymmetric, weighted Sobolev spaces, de Rham cohomology, Hodge Laplacian, mixed finite elements

1. Introduction

Suppose one has a problem defined on a three dimensional (3D) domain $\tilde{\Omega} \subset \mathbb{R}^3$ that is symmetric with respect to the $z$-axis. In other words, $\Omega \subset \mathbb{R}^3$ is obtained by rotating a two dimensional (2D) domain $\Omega \subset \mathbb{R}^2$ with respect to the axis of symmetry. Throughout the paper, we will assume that $\Omega$ is a bounded Lipschitz domain. If additionally the data function is independent of the angular variable $\theta$ when written in terms of cylindrical coordinates (and the cylindrical basis for vector-valued functions), then the solution to the problem is also independent of the $\theta$-variable (see Bernardi et al. (1999)), and the 3D problem can be reduced to a 2D one. This is an attractive feature, since the discrete problem corresponding to the 2D problem is significantly smaller than that corresponding to the 3D problem, but due to the Jacobian arising from change of variables, once the dimension reduction is done one is left in weighted Sobolev spaces where the weight function is the radial component $r$.

When $\tilde{\Omega}$ contains the axis of symmetry, and therefore the corresponding 2D domain $\Omega$ has nonzero intersection with the axis $r = 0$, the space of square integrable functions over $\Omega$ is strictly included in the space of functions that are square integrable with the weight function $r$ over $\Omega$. Therefore the analysis of the 2D problem in weighted spaces is not a trivial one.

There are several weighted Sobolev spaces that arise naturally from axisymmetric problems. First of all, let $L^2(\tilde{\Omega})$ and $L^2(\Omega)$ be the spaces of square integrable functions on $\tilde{\Omega}$ (scalar-valued and vector-valued respectively). These are Hilbert spaces with the inner product

\[ (u,v)_{L^2(\tilde{\Omega})} = \int_{\tilde{\Omega}} uv dV. \]

We will write $(\cdot,\cdot)$ to denote this $L^2$ inner product. Next, define $L^2(\tilde{\Omega})$ and $\tilde{L}^2(\tilde{\Omega})$ by the closed subspaces of $L^2(\tilde{\Omega})$ and $L^2(\tilde{\Omega})$ respectively that consist of functions that are axisymmetric in the following

†Email: ohmx@jmu.edu

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2 of 15

sense:

\[ \tilde{L}^2(\hat{\Omega}) = \\left\{ u \in L^2(\hat{\Omega}) : \frac{\partial u}{\partial \theta} = 0 \right\}, \]

\[ L^2(\hat{\Omega}) = \\left\{ u, e_r + u_\theta e_\theta + u_z e_z \in \tilde{L}^2(\hat{\Omega}) : \frac{\partial u_r}{\partial \theta} = \frac{\partial u_\theta}{\partial \theta} = \frac{\partial u_z}{\partial \theta} = 0 \right\}, \]

where \( e_r, e_\theta, \) and \( e_z \) denote the cylindrical basis of \( \mathbb{R}^3 \).

Throughout this paper, we will use the \( \hat{\cdot} \) symbol above a space of functions or differential forms to denote its axisymmetric subspace. If \( \hat{\cdot} \) is written above a function or a differential form, that indicates that it is a function or a differential form that is independent of the \( \theta \)-variable. When we want to make the connection between an axisymmetric function defined on \( \hat{\Omega} \) and the corresponding function defined on \( \Omega \), we will also use this notation. In other words, \( \hat{f} \) defined on \( \hat{\Omega} \) and \( f \) defined on \( \Omega \) are connected through the equation

\[ \hat{f}(r, \theta, z) = f(r, z). \]

Similar notation will be used to denote an “axisymmetric” differential \( k \)-form defined on \( \hat{\Omega} \) and the corresponding differential \( k \)-form defined on \( \Omega \), which we will discuss in the next section. Notice that any function \( f \in L^2(\hat{\Omega}) \) will satisfy

\[ 2\pi \int_{\Omega} f^2rdrdz = \int_{\Omega} f^2dV < \infty. \]

Therefore, \( \hat{f} \) is in \( L^2(\hat{\Omega}) \) if and only if \( f \) is in \( L^2(\Omega) \), where

\[ L^2(\Omega) = \left\{ u : \int_{\Omega} u^2rdrdz < \infty \right\}. \]

\( L^2(\Omega) \) is a Hilbert space with the inner product being

\[ (u,v)_{L^2(\Omega)} = \int_{\Omega} uvrdrdz. \]

We will write \( (\cdot,\cdot) \), to denote this \( L^2 \) inner product, and the induced norm will be denoted by \( \| \cdot \|_r \). This is how weighted spaces arise from axisymmetric problems. Similarly, there is a weighted space corresponding to \( \tilde{H}^1(\hat{\Omega}) \), the space of functions in \( L^2(\hat{\Omega}) \) whose gradient is also in \( \tilde{L}^2(\hat{\Omega}) \), and \( H(\text{curl},\hat{\Omega}) \), the space of functions in \( \tilde{L}^2(\hat{\Omega}) \) whose curl-value is also in \( \tilde{L}^2(\hat{\Omega}) \). Boundary conditions on \( \partial \hat{\Omega} \) transfers into boundary conditions on \( \Gamma_1 \), where \( \Gamma_1 \) denotes the part of the boundary of \( \Omega \) that is not on the axis of symmetry. The relevant weighted spaces are described in section 5.

Many authors have been analyzing axisymmetric problems in these weighted function spaces and the corresponding weighted norms (Bernardi et al. (1999); Assous et al. (2002, 2003b,a); Ciarlet et al. (2006); Bellahcini et al. (2006); Copeland et al. (2008); Oh (2010); Gopalakrishnan & Oh (2012); Ciarlet & Labrunie (2011) to name just a few). Commuting diagrams in weighted function spaces have been constructed as well (see Copeland et al. (2008, 2010); Gopalakrishnan & Oh (2012)). To our knowledge, however, axisymmetric problems have not yet been studied through the use of differential forms and exterior calculus. Finite element exterior calculus is of growing importance and interest in understanding the finite element method. This connection was first recognized in Bossavit (1988),
where the equivalence between Whitney forms (see Whitney (1957)) and Nédélec elements (see Nédélec (1980)) was pointed out. Finite element exterior calculus was discussed in Hiptmair (1999, 2002) for electromagnetic problems, and was further developed and established by Arnold, Falk, and Winther (see Arnold (2002); Arnold et al. (2006a,b,c, 2010)). In this paper, we develop a framework that allows one to analyze axisymmetric problems by using differential forms and exterior calculus. We will construct “weighted” differential form spaces and the “weighted” $L^2$ de Rham complex. Then we apply the theory developed in Arnold et al. (2010) to get results in weighted spaces.

Here is the outline of the paper. In the next section, we introduce some basic notions from exterior calculus and define an “axisymmetric” differential $k$-form. In section 3, we will construct the “axisymmetric” $L^2$ de Rham complex, and by decoupling this complex we will construct the corresponding “weighted” $L^2$ de Rham complex on $\Omega \subset \mathbb{R}^3_+$. In section 4, we will get results (Hodge decomposition, Poincaré inequality, and the well-posedness of the mixed Hodge Laplacian problem) in weighted spaces by applying the theory in Arnold et al. (2010). In the following section, we use bounded cochain projections to achieve stability and convergence of the discrete weighted mixed Hodge Laplacian problem. Future work, which is the generalization of this work to problems defined on axisymmetric domains with general data (not necessarily axisymmetric), is described in the final section.

2. Preliminaries

In this section, we first summarize some basic theory of exterior calculus in $n$-dimensions. We then define “axisymmetric” differential $k$-forms in $\mathbb{R}^3$.

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$. Then, $\text{Alt}^k V$ is the space of real-valued alternating $k$-linear forms on $V$. An inner product on $V$ induces an inner product on $\text{Alt}^k V$ by

$$<\omega, \eta>_{\text{Alt}^k V} = \sum_{\sigma} \omega(e_{\sigma(1)}, \cdots, e_{\sigma(k)}) \eta(e_{\sigma(1)}, \cdots, e_{\sigma(k)}),$$

where the sum is over increasing sequences $\sigma : \{1, \cdots, k\} \to \{1, \cdots, n\}$, and $e_1, \cdots, e_n$ is any orthonormal basis of $V$. If $\omega \in \text{Alt}^k V$ and $\eta \in \text{Alt}^k V$, then the wedge product $\omega \wedge \eta \in \text{Alt}^{k+k} V$ is defined by

$$(\omega \wedge \eta)(v_1, v_2, \cdots, v_{j+k}) = \sum_{\sigma \in S_{j,k}} (\text{sign} \sigma) \omega(v_{\sigma(1)}, \cdots, v_{\sigma(j)}, v_{\sigma(j+1)}, \cdots, v_{\sigma(j+k)}),$$

where $S_{j,k}$ denotes the set of permutations of $\{1, \cdots, j+k\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(j)$ and $\sigma(j+1) < \sigma(j+2) < \cdots < \sigma(j+k)$. If we assign an orientation to $V$ (by assigning a positive orientation to some particular ordered basis), then we may define a unique volume form, denoted by $\text{vol}$, in $\text{Alt}^k V$, defined by

$$\text{vol}(e_1, e_2, \cdots, e_n) = 1,$$

for any positively oriented orthonormal basis $e_1, \cdots, e_n$ of $V$. The Hodge star operator $\ast : \text{Alt}^k V \to \text{Alt}^{n-k} V$ is defined by

$$\omega \wedge \mu = <\ast \omega, \mu>_{\text{Alt}^{n-k} V} \text{vol},$$

for all $\omega \in \text{Alt}^k V, \mu \in \text{Alt}^{n-k} V$.

Throughout this paper, $\tilde{\Omega}$ denotes the axisymmetric domain in $\mathbb{R}^3$, and $\Omega$ denotes the corresponding 2D domain in $\mathbb{R}^2_+$. A differential $k$-form $\omega$ on $\tilde{\Omega}$ is a map that assigns to each $x \in \tilde{\Omega}$ an element in...
with the weight words, metric differential for all axisymmetric if and only if.

We will write differential forms by using cylindrical coordinates. In other words, we will use differential forms by using cylindrical coordinates so if $dx$ is the one-form which associates to a vector its $j$th coordinate. We say that $\omega \in \Lambda^k(\Omega)$ if $\alpha_{\sigma} \in C^\infty(\Omega)$ for all $\sigma$. In other words, $\Lambda^k(\Omega)$ is the space of smooth differential $k$-forms. When the domain is clear from the context, we will write $\Lambda^k$ to denote this space.

The exterior derivative applied to a $k$-form that is written by using standard coordinates is

$$d(\alpha_{\sigma} dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}) = \sum_{j=1}^3 \frac{\partial \alpha_{\sigma}}{\partial x_j} dx_j \wedge dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)},$$

so if $\omega$ is of the form (2.1), then $d\omega$ is the sum of (2.2) over all $\sigma$. In this paper, however, we will write differential forms by using cylindrical coordinates $(r, \theta, z)$ since we are interested in performing dimension reduction by using cylindrical coordinates. In other words, we will use $dr, rd\theta,$ and $dz$ instead of $dx_1, dx_2,$ and $dx_3$ when writing a $k$-form. The exterior derivative applied to a $k$-form written in this way can be obtained by using (2.2) and the relations

$$dx_1 = \cos \theta dr - r \sin \theta d\theta,$$
$$dx_2 = \sin \theta dr + r \cos \theta d\theta,$$
$$dx_3 = dz,$$

and

$$dr = \cos \theta dx_1 + \sin \theta dx_2,$$
$$rd\theta = -\sin \theta dx_1 + \cos \theta dx_2,$$
$$dz = dx_3.$$  

We will call a differential $k$-form axisymmetric when each of its components are invariant under rotation. For example, a 1-form $\omega = f dr + g d\theta + hdz$ and a 2-form $\eta = frd\theta \wedge dz + gdz \wedge dr + hdr \wedge r d\theta$ are axisymmetric if and only if $f, g,$ and $h$ are functions that are invariant under rotation (independent of the $\theta$-variable). A 0-form $f$ and a 3-form $frd\theta \wedge dz$ are axisymmetric if and only if $f$ is a function that is invariant under rotation. We will write $\tilde{\cdot}$ to indicate that a $k$-form is axisymmetric. Now, $L^2\Lambda^k(\Omega)$ is the space of differential $k$-forms with each component a square integrable function. This is a Hilbert space with the inner product being

$$<\omega, \eta>_{L^2\Lambda^k} = \int_\Omega <\omega, \eta>_{\Lambda^k T_3(\Omega)} \, dV,$$

for all $\omega, \eta \in L^2\Lambda^k(\Omega)$. Define $L^2\Lambda^k(\Omega)$ to be the closed subspace of $L^2\Lambda^k(\Omega)$ that consists of axisymmetric differential $k$-forms. Similarly, we can define a weighted space of differential $k$-forms. In other words, $L^2\Lambda^k(\Omega)$ consists of differential $k$-forms ($k = 0, 1, 2$) whose each component is square integrable with the weight $r$ (with the measure $rd\theta dz$). This is again a Hilbert space with the inner product

$$<\omega, \eta>_{L^2\Lambda^k} = \int_\Omega <\omega, \eta>_{\Lambda^k T_3(\Omega)} \, r dz,$$
for all \( \omega, \eta \in L^2_\Omega(\Omega) \). The induced norm is denoted by \( \| \cdot \|_{L^2_\Omega} \). As there is a natural isomorphism between \( L^2_\Omega(\hat{\Omega}) \) and \( L^2_\Omega(\Omega) \) (see section 1), there is also a natural isomorphism between \( L^2_\Omega(\hat{\Omega}) \) and \( L^2_\Omega(\Omega) \). When denoting the space of differential forms, \( \hat{\Omega} \) or \( \Omega \) may be omitted when the domain is clear from the context. We will omit \( L^2_\Omega(\hat{\Omega}) \) and write \( < \cdot, \cdot > \) to denote the inner product in \( L^2_\Omega(\hat{\Omega}) \). Similarly, we will write \( < \cdot, \cdot > \) to denote the inner product in \( L^2_\Omega(k) \).

All differential \( k \)-forms appearing in the remainder of this section are forms that are defined on \( \hat{\Omega} \) unless stated otherwise. We have the well-known \( L^2 \) de Rham complex (Arnold et al., 2010, Section 4):

\[
0 \rightarrow H^0 \overset{d}{\rightarrow} H^1 \overset{d}{\rightarrow} H^2 \overset{d}{\rightarrow} H^3 \rightarrow 0,
\]

where \( d \) is the exterior derivative, and \( H^k \) denotes the space of differential \( k \) forms in \( L^2(\hat{\Omega}) \) whose exterior derivative is in \( L^2(\hat{\Omega})^{k+1} \). With the inner product

\[
< \omega, \eta >_{H^k} = < \omega, \eta > + < d\omega, d\eta >,
\]

\( H^k \) is a Hilbert space.

Suppose \( \Omega_1 \subset \mathbb{R}^n \) and \( \Omega_2 \subset \mathbb{R}^m \) are smooth manifolds. Then, a smooth map \( \phi : \Omega_1 \rightarrow \Omega_2 \) induces a pullback \( \phi^* : \Lambda^k(\Omega_2) \rightarrow \Lambda^k(\Omega_1) \):

\[
(\phi^* \omega)_x(v_1, \ldots, v_k) = \omega_{\phi(x)}(D\phi_x(v_1), \ldots, D\phi_x(v_k)),
\]

for \( x \in \Omega_1 \) and \( v_1, \ldots, v_k \in T_x \Omega_1 \), where the linear map \( D\phi_x : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is the derivative of \( \phi \) at \( x \). Thus, we can let \( \phi : \partial \hat{\Omega} \rightarrow \hat{\Omega} \) be the inclusion map and consider its pullback. This is called the trace map, and it will be denoted by \( \text{tr} : \Lambda^k(\hat{\Omega}) \rightarrow \Lambda^k(\partial \hat{\Omega}) \). The map \( \omega \mapsto \text{tr} \omega \) is continuous from \( H^k(\hat{\Omega}) \) to \( H^{-1/2}(\partial \hat{\Omega}) \) (Arnold et al., 2006c, Section 2). Therefore,

\[
H^k(\hat{\Omega}) = \left\{ \omega \in H^k(\hat{\Omega}) : \text{tr} \omega = 0 \right\}
\]

is well-defined, and furthermore, one can construct the \( L^2 \) de Rham complex with boundary conditions:

\[
0 \rightarrow H^0 \overset{d}{\rightarrow} H^1 \overset{d}{\rightarrow} H^2 \overset{d}{\rightarrow} H^3 \rightarrow 0.
\]  \( (2.5) \)

Before we end this section, let us examine the integration by parts formula for differential forms. By the Leibniz rule, Stokes theorem, and the fact that pullbacks respect the wedge product, we obtain the integration by parts formula for differential forms:

\[
\int_{\hat{\Omega}} d\omega \wedge \eta = (-1)^k \int_{\hat{\Omega}} \omega \wedge d\eta + \int_{\partial \hat{\Omega}} \text{tr} \omega \wedge \text{tr} \eta, \quad \text{for all } \omega \in \Lambda^{k-1}, \eta \in \Lambda^{n-k}. \]  \( (2.6) \)

On \( \hat{\Omega} \), the Hodge star operator * from \( \Lambda^k \) to \( \Lambda^{n-k} \) satisfies

\[
\int_{\hat{\Omega}} \omega \wedge \mu = < *\omega, \mu >_{L^2(\Lambda^{n-k})},
\]

for all \( \mu \in \Lambda^{n-k} \). The coderivative operator \( \delta \) maps a \( k \)-form to a \((k-1)\)-form in the following way:

\[
*(\delta \omega) = (-1)^k d(*\omega).
\]
In analogy with the space $H^k$, we define the space $H^*\Lambda^k$ by

$$H^*\Lambda^k = \{ \omega \in L^2\Lambda^k : \delta \omega \in L^2\Lambda^{k-1} \}.$$ 

Setting $\eta = \ast \mu$ in (2.6), we obtain another integration by parts formula for differential forms that includes both the exterior derivative and the coderivative:

$$<d\omega, \mu> = <\omega, \delta \mu> + \int_{\partial\Omega} \text{tr}\omega \wedge \text{tr}\ast \mu, \quad \text{for all} \ \omega \in \Lambda^{k-1}, \mu \in \Lambda^k.$$ 

This can be extended to

$$<d\omega, \mu> = <\omega, \delta \mu>, \quad \text{for all} \ \omega \in H_\circ \Lambda^{k-1}, \mu \in H^*\Lambda^k. \quad (2.7)$$

3. A Weighted $L^2$ de Rham Complex Arising from the Axisymmetric $L^2$ de Rham Complex

It is well-known that when the domain is in $\mathbb{R}^3$ the exterior derivative applied to a one, two, and three form is the gradient, curl, and divergence operator respectively. In this section, we interpret the exterior derivative $d$ and the coderivative $\delta$ when applied to differential forms on $\Omega \subset \mathbb{R}^3$ under axial symmetry, i.e., $\frac{\partial}{\partial \theta} = 0$. We will write $\partial$ to denote partial derivatives, $\partial_r = \frac{\partial}{\partial r}$, etc.

By using (2.2), (2.3), (2.4), and the fact that $\partial_\theta \tilde{f} = \partial_\theta \tilde{g} = \partial_\theta \tilde{h} = 0$, we get

$$d\tilde{f} = \partial_r \tilde{f} dr + \partial_\theta \tilde{f} d\theta,$$

$$d(\tilde{f}dr + \tilde{g}r d\theta + \tilde{h}dz) = (-\partial_\theta \tilde{g})rd\theta \wedge dz + (\partial_r \tilde{f} - \partial_\theta \tilde{h})dz \wedge dr + \frac{1}{r} \partial_r (r\tilde{g})dr \wedge r\theta,$$

$$d(\tilde{f}r \theta \wedge dz + \tilde{g}d\theta \wedge dr + \tilde{h}dr \wedge r\theta) = (-\partial_r (r\tilde{f}) + \partial_\theta \tilde{h})dr \wedge r\theta \wedge dz. \quad (3.1)$$

This is closely related to the grad, curl, and div formula in cylindrical coordinates under axial symmetry.

Notice that the exterior derivative under axial symmetry decouples in the following sense. For example,

$$d(\tilde{f}dr + \tilde{g}r d\theta + \tilde{h}dz) = (-\partial_\theta \tilde{g})rd\theta \wedge dz + (\partial_r \tilde{f} - \partial_\theta \tilde{h})dz \wedge dr + \frac{1}{r} \partial_r (r\tilde{g})dr \wedge r\theta,$$

the $dz \wedge dr$ component is only affected by $\tilde{f}$ and $\tilde{h}$, while the $dr \wedge r\theta$ and $rd\theta \wedge dz$ components are only affected by $\tilde{g}$. In other words,

$$d(\tilde{f}dr + 0rd\theta + \tilde{h}dz) = 0rd\theta \wedge dz + (\partial_r \tilde{f} - \partial_\theta \tilde{h})dz \wedge dr + 0dr \wedge r\theta,$$

$$d(0dr + \tilde{g}d\theta + 0dz) = (-\partial_\theta \tilde{g})rd\theta \wedge dz + 0dz \wedge dr + \frac{1}{r} \partial_r (r\tilde{g})dr \wedge r\theta. \quad (3.2)$$

Similarly,

$$d(\tilde{f}r \theta \wedge dz + 0dz \wedge dr + \tilde{h}dr \wedge r\theta) = \frac{1}{r} \partial_r (r\tilde{f}) + \partial_\theta \tilde{h})dr \wedge r\theta \wedge dz,$$

$$d(0rd\theta \wedge dz + \tilde{g}dz \wedge dr + 0dr \wedge r\theta) = 0dr \wedge r\theta \wedge dz. \quad (3.3)$$

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Keeping this in mind, let us define the following subspaces of $\bar{H}_\circ \Lambda^1$ and $\bar{H}_\circ \Lambda^2$.

\[
\begin{align*}
\bar{H}_\circ^M \Lambda^1 &= \{ fdr + \bar{g}rd\theta + \bar{h}dz \in \bar{H}_\circ \Lambda^1 : \bar{g} = 0 \}, \\
\bar{H}_\circ^A \Lambda^1 &= \{ fdr + \bar{g}rd\theta + \bar{h}dz \in \bar{H}_\circ \Lambda^1 : \bar{f} = 0 \text{ and } \bar{h} = 0 \}, \\
\bar{H}_\circ^M \Lambda^2 &= \{ frd\theta \wedge dz + \bar{g}dz \wedge dr + \bar{h}dr \wedge rd\theta \in \bar{H}_\circ \Lambda^2 : \bar{g} = 0 \}, \\
\bar{H}_\circ^A \Lambda^2 &= \{ frd\theta \wedge dz + \bar{g}dz \wedge dr + \bar{h}dr \wedge rd\theta \in \bar{H}_\circ \Lambda^2 : \bar{f} = 0, \bar{h} = 0 \},
\end{align*}
\]  

(3.4)

Now, let us consider the axisymmetric $L^2$ de Rham complex with homogeneous boundary conditions:

\[
0 \rightarrow \bar{H}_\circ \Lambda^0 \xrightarrow{d} \bar{H}_\circ \Lambda^1 \xrightarrow{d} \bar{H}_\circ \Lambda^2 \xrightarrow{d} \bar{H}_\circ \Lambda^3 \rightarrow 0. 
\]  

(3.5)

This is a subcomplex of (2.5). Due to (3.2) and (3.3), the axisymmetric $L^2$-de Rham complex with boundary conditions (3.5) decouples into the following two de Rham complexes:

\[
0 \rightarrow \bar{H}_\circ \Lambda^0 \xrightarrow{d} \bar{H}_\circ^M \Lambda^1 \xrightarrow{d} \bar{H}_\circ^A \Lambda^2 \rightarrow 0
\]  

(3.6)

and

\[
0 \rightarrow \bar{H}_\circ^A \Lambda^1 \xrightarrow{d} \bar{H}_\circ^M \Lambda^2 \xrightarrow{d} \bar{H}_\circ \Lambda^3 \rightarrow 0.
\]  

(3.7)

Note that, due to (3.3), all two forms in $\bar{H}_\circ^A \Lambda^2$ get sent to zero by the exterior derivative under axial symmetry. In this paper, we will focus on the axisymmetric de Rham complex (3.6).

Next, define $\tilde{d}$ as the two-dimensional operator corresponding to the exterior derivative $d$ under axial symmetry:

\[
\tilde{d}f = \partial_r f dr + \partial_z f dz, \\
\tilde{d}(fdr + hdz) = (\partial_r f - \partial_z h)dz \wedge dr.
\]

Notice that $\tilde{d}$ is defined on differential forms whose domain is the two-dimensional domain $\Omega$. Define

\[
H_r \Lambda^k = H_r \Lambda^k(\Omega) = \{ \omega \in L^2_r \Lambda^k : \tilde{d} \omega \in L^2_r \Lambda^{k+1} \}.
\]

This is a Hilbert space with the inner product being

\[
< \omega, \nu >_{H_r \Lambda^k} = < \omega, \nu >_r + < \tilde{d} \omega, \tilde{d} \nu >_r.
\]

The induced norm is denoted by $\| - \|_{H_r \Lambda^k}$. We can also define a closed subspace of $H_r \Lambda^k$ with boundary conditions:

\[
H_{\text{r,v}} \Lambda^k = \{ \omega \in H_r \Lambda^k : \text{tr} \tilde{\omega} = 0 \}. 
\]  

(3.8)

Let us make the connection between $\omega \in L^2_r \Lambda^k(\Omega)$ and $\tilde{\omega} \in L^2_r \Lambda^k(\tilde{\Omega})$ clear here:

\[
\omega = f dr + hdz, \\
\tilde{\omega} = \tilde{f} dr + 0 rd\theta + \tilde{h}dz,
\]

and

\[
\omega = f dz \wedge dr, \\
\tilde{\omega} = 0 rd r + dz \wedge dr + 0 dr \wedge rd\theta.
\]
If \( \omega \in L^2_\Sigma \Lambda^0 \), then \( \omega \) is a function in \( L^2_\Sigma (\Omega) \), and \( \bar{\omega} \) has the expected meaning explained in section 2.

Therefore, by definition (3.8), differential forms in \( H_{r;0} \Lambda^k \) have zero boundary conditions on \( \Gamma_1 \). Recall that \( \Gamma_1 \) is the part of \( \partial \Omega \) that is not on the axis \( r = 0 \).

Then, instead of analyzing the axisymmetric de Rham complex (3.6), we can analyze the following weighted de Rham complex whose domain is \( \Omega \subset \mathbb{R}^2_+ \):

\[
0 \rightarrow H_{r;0} \Lambda^0 \xrightarrow{d} H_{r;0} \Lambda^1 \xrightarrow{d} H_{r;0} \Lambda^2 \rightarrow 0.
\]

(3.9)

Now, let us construct the dual complex of (3.9) by considering the coderivative \( \delta \) of axisymmetric differential forms. Under axial symmetry, we have that

\[
\delta(f dr + \bar{g} rd\theta + \bar{h} dz) = -\frac{1}{r} \partial_r (f \bar{r}) - \partial_z \bar{h},
\]

\[
\delta(f rd\theta \wedge dz + \bar{g} dz \wedge dr + \bar{h} dr \wedge rd\theta) = -\partial_d \bar{g} dr + (\partial_f \bar{f} - \partial_h \bar{h}) rd\theta + \frac{1}{r} \partial_r (\bar{g} r) dz,
\]

\[
\delta(f dr \wedge rd\theta \wedge dz) = -\partial_q f rd\theta \wedge dz + 0d\theta \wedge dr - \partial_z f dr \wedge rd\theta.
\]

(3.10)

Like we did for the exterior derivative, define \( \tilde{\delta} \) as the corresponding 2D operator of the coderivative \( \delta \) under axial symmetry, i.e.,

\[
\tilde{\delta}(f dr + \bar{h} dz) = -\frac{1}{r} \partial_r (f \bar{r}) - \partial_z \bar{h},
\]

\[
\tilde{\delta}(g dz \wedge dr) = -\partial_d g dr + \frac{1}{r} \partial_r (rg) dz.
\]

Now we are ready to construct the dual complex of the weighted \( L^2 \) complex (3.9). We write \( H^*_{r;0} \Lambda^k \) to denote the space of differential forms such that \( \omega \in H^*_{r;0} \Lambda^k \) if and only if \( \omega \in L^2_\Sigma \Lambda^k \) and \( \bar{\omega} \in L^2_\Sigma \Lambda^{k-1} \).

THEOREM 3.1 Let \( \tilde{d} \) be the exterior derivative viewed as an unbounded operator \( L^2_\Sigma \Lambda^{k-1} \rightarrow L^2_\Sigma \Lambda^k \) with domain \( H_{r;0} \Lambda^{k-1} \). Then the adjoint of \( \tilde{d} \), as an unbounded operator \( L^2_\Sigma \Lambda^k \rightarrow L^2_\Sigma \Lambda^{k-1} \), has \( H^*_{r;0} \Lambda^k \) as its domain and coincides with the operator \( \tilde{\delta} \).

Proof. We must show that for \( \mu \in L^2_\Sigma \Lambda^k \), there exists \( \omega \in L^2_\Sigma \Lambda^{k-1} \) such that

\[
< \mu, \tilde{d} \bar{v} > = < \omega, v >, \text{ for all } v \in H_{r;0} \Lambda^{k-1}
\]

(3.11)

if and only if \( \mu \in H^*_{r;0} \Lambda^k \) and \( \omega = \tilde{\delta} \mu \). Since

\[
< d\bar{v}, \tilde{\mu} > = < \bar{v}, \delta \tilde{\mu} >, \text{ for all } \bar{v} \in H_{r} \Lambda^{k-1}, \tilde{\mu} \in H^*_{r} \Lambda^k,
\]

by the integration by parts formula (2.7), we have that

\[
< \tilde{d} \bar{v}, \mu > = < v, \tilde{\delta} \mu >, \text{ for all } v \in H_{r;0} \Lambda^{k-1}, \mu \in H^*_{r;0} \Lambda^k.
\]

Conversely, if (3.11) is true, then

\[
< \tilde{\mu}, d\bar{v} > = < \bar{\omega}, \bar{v} >, \text{ for all } \bar{v} \in H_{r} \Lambda^{k-1}.
\]

Thus, \( \tilde{\mu} \) has a weak exterior coderivative (see Arnold et al., 2010, p.323)) in \( L^2 \Lambda^{k-1} \), namely \( \bar{\omega} = \delta \tilde{\mu} \).

Therefore, \( \tilde{\mu} \in H^*_{r} \Lambda^k \), and so \( \mu \in H^*_{r;0} \Lambda^k \). This completes the proof. \( \square \)
In the remainder of the paper, the domain of all differential forms and functions is $\Omega \subset \mathbb{R}^2$. Notice that $\tilde{d}$ is a closed, densely defined operator on $L^2_\Omega \Lambda^k$ such that the range of $\tilde{d}$ is in $L^2_\Omega \Lambda^{k+1}$, and $\tilde{d} \circ \tilde{d} = 0$. In other words,
\[ 0 \to L^2_\Omega \Lambda^0 \xrightarrow{\tilde{d}} L^2_\Omega \Lambda^1 \xrightarrow{\tilde{d}} L^2_\Omega \Lambda^{2} \to 0, \]
denoted by $(L^2_\Omega \Lambda^k, \tilde{d})$ is a (unbounded) Hilbert complex. It is also important to observe that the Hilbert complex $(L^2_\Omega \Lambda^k, \tilde{d})$ satisfies the compactness property (see Arnold et al. (2010)). In other words, $H_{r_o} \Lambda^k \cap H^*_\Lambda^k$ is dense in $L^2_\Omega \Lambda^k$ and the inclusion is a compact operator ((Mercier & Raugel, 1982, Lemma 4.2). Assous et al. (2002)). Therefore, $(L^2_\Omega \Lambda^k, \tilde{d})$ is a closed Hilbert complex with domain complex (3.9).

4. The Hodge Decomposition, Poincaré Inequality, and the Hodge Laplacian in Weighted Spaces

The results in this section are an application of the theory developed in (Arnold et al., 2010, Section 3). Theorem 3.1 and the fact that $(L^2_\Omega \Lambda^k, \tilde{d})$ is a closed Hilbert complex with domain complex (3.9) as discussed at the end of section 3 allow us to achieve the results. These results are interesting because they involve weighted norms, which are the key ingredients when analyzing axisymmetric problems.

In this section, we will write $\tilde{d}^k : H_{r_o} \Lambda^k \to H_{r_o} \Lambda^{k+1}$ and $\tilde{\delta}^k : H^*_\Lambda^k \to H^*_\Lambda^{k-1}$ with the superscript $k$ when it adds clarity to the context.

Let $\tilde{S}_{\Lambda, r_o} = S_{\Lambda, r_o}(L^2_\Omega \Lambda^k, \tilde{d}^k)$ denote the null space of $\tilde{d}^k$, and let $\tilde{B}_{\Lambda, r_o} = B_{\Lambda, r_o}(L^2_\Omega \Lambda^k, \tilde{d}^k)$ denote the range of $\tilde{d}^{k-1}$. The space of weighted harmonic forms, denoted by $\tilde{S}_{\Lambda, r_o}^\perp$, is the orthogonal complement of $\tilde{B}_{\Lambda, r_o}$ in $\tilde{S}_{\Lambda, r_o}$ with respect to the $\langle \cdot, \cdot \rangle_r$-inner product, which we will write as $\tilde{S}_{\Lambda, r_o}^\perp = \tilde{S}_{\Lambda, r_o} \cap \tilde{B}_{\Lambda, r_o}^\perp$. Let $\tilde{S}_{\Lambda, r_o}^*$ denote the null space of $\tilde{\delta}^k$, the dual operator of $\tilde{d}^k$ by Theorem 3.1, and let $\tilde{B}_{\Lambda, r_o}^*$ denote the range of $\tilde{\delta}^{k+1}$. Then, $\tilde{S}_{\Lambda, r_o} = \tilde{S}_{\Lambda, r_o}^\perp \cap \tilde{S}_{\Lambda, r_o}^*$, i.e.,

\[
\tilde{S}_{\Lambda, r_o}^\perp = \left\{ p \in H_{r_o} \Lambda^k \cap H^*_\Lambda^k : \tilde{d}^k u = 0, \tilde{\delta}^k u = 0 \right\}.
\]

We also have the weighted Hodge decomposition for the original complex $(L^2_\Omega \Lambda^k, \tilde{d})$ and the domain complex (3.9):

\[
L^2_\Omega \Lambda^k = \tilde{B}_{\Lambda, r_o} \oplus \tilde{S}_{\Lambda, r_o}^\perp \oplus \tilde{S}_{\Lambda, r_o}^\perp \oplus \tilde{S}_{\Lambda, r_o}^* = \tilde{B}_{\Lambda, r_o} \oplus \tilde{S}_{\Lambda, r_o}^\perp \oplus \tilde{B}_{\Lambda, r_o}^*,
\]

and

\[
H_{r_o} \Lambda^k = \tilde{B}_{\Lambda, r_o} \oplus \tilde{S}_{\Lambda, r_o}^\perp \oplus \tilde{S}_{\Lambda, r_o}^* = \tilde{B}_{\Lambda, r_o} \oplus \tilde{S}_{\Lambda, r_o}^\perp \oplus \tilde{S}_{\Lambda, r_o}^*,
\]

where $\tilde{S}_{\Lambda, r_o}^\perp = \tilde{S}_{\Lambda, r_o}^\perp \cap H_{r_o} \Lambda^k$.

Furthermore, we have the following weighted Poincaré inequality

\[
\|v\|_{H^1 \Lambda^k} \leq c_P \left\| \tilde{d}^k v \right\|_{L^2_\Omega \Lambda^k}, \quad \text{for all } v \in \tilde{S}_{\Lambda, r_o}^\perp.
\]

Next, we can also consider the so-called Hodge Laplacian for the weighted $L^2$ de Rham complex (3.9), i.e., $L = \tilde{d} \tilde{\delta} + \tilde{\delta} \tilde{d}$. This is an operator that appears after one performs dimension reduction to the ordinary Hodge Laplacian under axial symmetry, and its domain is

\[
D_L = \left\{ \omega \in H_{r_o} \Lambda^k \cap H^*_\Lambda^k : \tilde{d} \omega \in H^*_\Lambda^{k+1}, \tilde{\delta} \omega \in H_{r_o} \Lambda^{k-1} \right\}.
\]
The weighted mixed formulation of the Hodge Laplacian is stated in the following way. Find \((\sigma, u, p) \in H_{r,c}^{k-1} \times H_{r,c}^k \times \mathcal{S}^{k}_{r,c}\) such that

\[
\begin{aligned}
\langle \sigma, \tau \rangle_r - \langle \partial \tau, u \rangle_r &= 0, \quad \text{for all } \tau \in H_{r,c}^{k-1}, \\
\langle \partial \sigma, v \rangle_r + \langle \partial_R u, \partial_R v \rangle_r + \langle v, p \rangle_r &= \langle f, v \rangle_r, \quad \text{for all } v \in H_{r,c}^k, \\
\langle u, q \rangle_r &= 0, \quad \text{for all } q \in \mathcal{S}^{k}_{r,c}.
\end{aligned}
\] (4.4)

The following theorem follows from (Arnold et al., 2010, Theorem 3.1).

**Theorem 4.1** The weighted mixed formulation (4.4) is well-posed. Moreover,

\[
\|\sigma\|_{H_{r,c}^{k-1}} + \|u\|_{H_{r,c}^k} + \|p\|_{L^2_{r,c}^k} \leq c \|f\|_{L^2_{r,c}^k},
\]

where \(c\) is a constant depending only on the Poincaré constant \(c_p\) in (4.3).

Now let us interpret (4.4) in more concrete terms for \(k = 0, 1,\) and \(2\). Recall that \(\Gamma_1\) denotes the part of \(\partial \Omega\) that is not on the axis of rotation.

- **The Weighted Hodge Laplacian for \(k = 0\)**

\[
\begin{aligned}
-\text{div}_c \text{grad}_c u &= f \text{ in } \Omega, \\
\text{div}_c u &= 0 \text{ on } \Gamma_1,
\end{aligned}
\]

where \(\text{grad}_c v = (\partial_r v, \partial_z v)\) and \(\text{div}_c v = \partial_r v + \frac{1}{r} \partial_z (rv)_r\). This is the problem that one obtains after performing dimension reduction on the scalar Laplacian problem using axial symmetry.

- **The Weighted Hodge Laplacian for \(k = 1\)**

\[
\begin{aligned}
\sigma &= -\text{div}_c u \text{ in } \Omega, \\
\text{grad}_c \sigma + \text{curl}_c \text{curl}_c u &= f - p \text{ in } \Omega, \\
\sigma &= 0, \quad u \cdot t = 0 \text{ on } \Gamma_1, \\
p \in \mathcal{S}^1_{r,c}, \\
(u, q)_r &= 0 \text{ for all } q \in \mathcal{S}^1_{r,c},
\end{aligned}
\]

where \(\text{curl}_c (w_r, w_z) = \partial_z w_r - \partial_r w_z\), \(\text{curl}_c v = (-\partial_v \frac{1}{r} \partial_z (rv)_r)\), and \(t\) denotes the unit tangent vector oriented counter-clockwise. By finding the solution \(u = (u_r, u_z)\) to the above problem, one obtains the solution \(\bar{u} = (\bar{u}_r, 0, \bar{u}_z)\) to the meridian problem of the axisymmetric vector Laplacian.
• The Weighted Hodge Laplacian for $k = 2$

\[
\begin{align*}
\sigma &= \text{curl}_r u & \text{in } \Omega, \\
\text{curl}_r \sigma &= f - p & \text{in } \Omega, \\
\sigma \cdot t &= 0 & \text{on } \Gamma_1, \\
p &\in \mathcal{S}^{2}_{r,0}, \\
(u, q)_r &= 0 & \text{for all } q \in \mathcal{S}^{2}_{r,0}.
\end{align*}
\]

This curl$_r$, curl$_r$ problem is closely related to the azimuthal problem of the Maxwell equations under axial symmetry.

5. Stability and Convergence of the Weighted Mixed Method

Now, let us discuss the finite element approximation of (4.4). We will verify that the discrete mixed problem is stable and convergent by using bounded cochain projections constructed in Gopalakrishnan & Oh (2012) and by once again applying the theory in Arnold et al. (2010).

Define the following weighted Sobolev spaces

\[
\begin{align*}
H^1_r(\Omega) &= \{ v \in L^2_r(\Omega) : \text{grad}_r v \in L^2(\Omega) \}, \\
H_r(\text{curl}, \Omega) &= \{ w \in L^2_r(\Omega) : \text{curl}_r w \in L^2(\Omega) \}.
\end{align*}
\]

It is well-known that functions in $H^1_r(\Omega)$ have traces in $L^2_r(\Gamma_1)$ (Kufner et al. (1977)). Also, since $\partial \Omega$ is Lipschitz, the tangential trace operator on $H_r(\text{curl}, \Omega)$, denoted by $w \cdot t|_{\Gamma_1}$, is proved in (Copeland et al., 2008, Proposition 2.2) to be well-defined. Therefore, we may define the following closed subspaces:

\[
\begin{align*}
H^1_{r,0}(\Omega) &= \{ v \in H^1_r(\Omega) : v|_{\Gamma_1} = 0 \}, \\
H_{r,0}(\text{curl}, \Omega) &= \{ w \in H_r(\text{curl}, \Omega) : w \cdot t|_{\Gamma_1} = 0 \}.
\end{align*}
\]

Then, we may interpret the weighted de Rham complex (3.9) as

\[
0 \to H^1_{r,0}(\Omega) \xrightarrow{\text{grad}_r} H_{r,0}(\text{curl}, \Omega) \xrightarrow{\text{curl}_r} L^2_r(\Omega) \to 0.
\]

In other words, $\tilde{\partial}^0$ and $\tilde{\partial}^1$ are $\text{grad}_r$ and $\text{curl}_r$ respectively, and the weighted differential form spaces can be replaced by the corresponding weighted function spaces.

In Gopalakrishnan & Oh (2012), bounded cochain projections from these weighted spaces were constructed. In particular, let

\[
\begin{align*}
\mathcal{N}_1 &= \{ (a - bz, c + br) : a, b, c \in \mathbb{R} \}, \\
\mathcal{P}_1 &= \{ c_0 + c_1 r + c_2 z : c_i \in \mathbb{R} \text{ for } i = 0, 1, 2 \}.
\end{align*}
\]

Assume that $\Omega$ is meshed by a finite element triangulation $\mathcal{T}_h$ satisfying the usual geometrical conformity conditions (see Ciarlet (1978)). For simplicity, we assume that $\mathcal{T}_h$ is quasiuniform with a representative meshsize $h$, i.e., that the ratio $h$/diam $K$ is uniformly bounded for all $K \in \mathcal{T}_h$. The finite
element spaces we shall use are

\[ X_h = \{ u \in H^1(\Omega) : u|_K \in P_1 \text{ for all } K \in \mathcal{T}_h \}, \]
\[ W_h = \{ v \in H^1(\Omega) : v|_K \in \mathcal{N}_1 \text{ for all } K \in \mathcal{T}_h \}, \]
\[ X_{h,0} = \{ v \in X_h : v|_{\Gamma_1} = 0 \}, \]
\[ W_{h,0} = \{ v \in W_h : v|_{\Gamma_1} = 0 \}, \]
\[ S_h = \{ u \in L^2(\Omega) : u|_K \text{ is constant for all } K \in \mathcal{T}_h \}. \]

In Gopalakrishnan & Oh (2012), \( \Pi^p_h, \Pi^c_h, \) and \( \Pi^0_h, \) the so-called commuting smoothed projections, that make the following diagram commute were constructed. These bounded cochain projections are uniformly bounded operators from \( L^2_0(\Omega) \) to \( L^2_0(\Omega) \) (or from \( L^2(\Omega) \) to \( L^2(\Omega) \)) with respect to the meshsize \( h. \)

\[
\begin{array}{c}
H^1_{r,0}(\Omega) \xrightarrow{\text{grad}_x} H_{r,0}(\text{curl}, \Omega) \xrightarrow{\text{curl}_x} L^2_0(\Omega) \\
\downarrow \Pi^p_h \downarrow \Pi^c_h \downarrow \Pi^0_h \\
X_{h,0} \xrightarrow{\text{grad}_x} W_{h,0} \xrightarrow{\text{curl}_x} S_h
\end{array}
\tag{5.1}
\]

Now, the subcomplex

\[ 0 \to X_{h,0} \xrightarrow{\text{grad}_x} W_{h,0} \xrightarrow{\text{curl}_x} S_h \to 0 \]

is again a closed Hilbert complex, and we may apply the theory of Arnold et al. (2010) like we did for the continuous case to obtain the weighted Hodge decomposition and the weighted Poincaré inequality for these discrete spaces. Note that while \( \tilde{d} \) for the subcomplex is just the restriction of \( \tilde{d} \) of the original complex, its adjoint is not the restriction of \( \tilde{\delta}. \)

The weighted mixed Hodge Laplacian problem (4.4) can be restated in weighted function spaces (instead of weighted differential form spaces) in the following way.

Find \((\sigma, u, p) \in V_{r,0}^{-1} \times V_{r,0}^k \times H^k_{r,0}\) such that

\[
(\sigma, \tau)_r - (\tilde{d}\tau, u)_r = 0, \text{ for all } \tau \in V_{r,0}^{-1},
\]
\[
(\tilde{d}\sigma, v)_r + (du, dv)_r + (v, p)_r = (f, v)_r, \text{ for all } v \in V_{r,0}^k,
\]
\[
(u, q)_r = 0, \text{ for all } q \in H^k_{r,0},
\]

where \( V_{r,0}^0 = H^1_{r,0}(\Omega), V_{r,0}^1 = H_{r,0}(\text{curl}, \Omega), \) and \( V_{r,0}^2 = L^2_0(\Omega). \) Also, if

\[
H_r(\text{div}, \Omega) = \{ u \in L^2_0(\Omega) : \text{div}_r u \in L^2_0(\Omega) \}, \]
\[
H_r(\text{curl}, \Omega) = \{ u \in L^2_0(\Omega) : \text{curl}_r u \in L^2_0(\Omega) \},
\]

then the spaces of harmonic forms are

\[
H^0_{r,0} = \{ v \in H^1_{r,0}(\Omega) : \text{grad}_x v = 0 \} = \{ 0 \},
\]
\[
H^1_{r,0} = \{ v \in H_{r,0}(\text{curl}, \Omega) \cap H_r(\text{div}, \Omega) : \text{curl}_x v = 0, \text{div}_x v = 0 \},
\]
\[
H^2_{r,0} = \{ v \in H_r(\text{curl}, \Omega) : \text{curl}_x v = 0 \}.
\]
Note that $H^2_{h,\circ}$ reduces to $\{0\}$ when $\Omega$ has non-empty intersection with the $r$ axis $r=0$. Next, the discrete weighted mixed Hodge Laplacian problem can be stated as follows.

Find $(\sigma_h, u_h, p_h) \in V^{k-1}_{h,\circ} \times V^k_{h,\circ} \times H^k_{h,\circ}$ such that

\[
\begin{align*}
(\sigma_h, \tau_h) - (\delta \tau_h, u_h)_r &= 0, \text{ for all } \tau_h \in V^{k-1}_{h,\circ}, \\
(\delta \sigma_h, v_h)_r + (du_h, dv_h)_r + (v_h, p_h)_r &= (f, v_h)_r, \text{ for all } v_h \in V^k_{h,\circ}, \\
(u_h, q_h)_r &= 0, \text{ for all } q \in H^k_{h,\circ},
\end{align*}
\]

(5.3)

where $V^0_{h,\circ} = X_{h,\circ}$, $V^1_{h,\circ} = W_{h,\circ}$ and $V^2_{h,\circ} = S_h$. $H^k_{h,\circ}$ denotes the space of discrete harmonic forms. In general, $H^k_{h,\circ}$ is not in $H^k_{r,\circ}$, but since $dV^k_{h,\circ} \subset V^{k+1}_{h,\circ}$ (subcomplex property) and there exist bounded cochain projections (5.1), $H^k_{h,\circ}$ provides a faithful approximation of $H^k_{r,\circ}$ (Arnold et al., 2010, Theorem 3.4, 3.5). The computational aspect of $H^k_{h,\circ}$ can be found in Amrouche et al. (1998) or (Arnold et al., 2010, Lemma 3.10).

Since the discrete complex is a subcomplex of the closed Hilbert complex (3.9) (and therefore also closed), we obtain the stability of the discrete mixed problem (5.3) by (Arnold et al., 2010, Theorem 3.1). Furthermore, we obtain the following error estimates.

**Theorem 5.1** Let $(\sigma, u, p) \in V^{k-1}_{r,\circ} \times V^k_{r,\circ} \times H^k_{r,\circ}$ be the solution of (5.2) and $(\sigma_h, u_h, p_h) \in V^{k-1}_{h,\circ} \times V^k_{h,\circ} \times H^k_{h,\circ}$ be the solution of (5.3). Then for some constant $C$ independent of $h$ and $(\sigma, u, p)$, we have

\[
\begin{align*}
\|d(\sigma - \sigma_h)\|_r &\leq C \inf_{v_h \in V^{k-1}_{h,\circ}} \|d\sigma - dv_h\|_r, \\
\|\sigma - \sigma_h\|_r &\leq C \inf_{v_h \in V^{k-1}_{h,\circ}} \|\sigma - v_h\|_r + \eta \inf_{v_h \in V^{k-1}_{h,\circ}} \|d\sigma - dv_h\|_r, \\
\|p - p_h\|_r &\leq C \inf_{q_h \in H^k_{h,\circ}} \|p - q_h\|_r + \mu \inf_{v_h \in V^{k-1}_{h,\circ}} \|d\sigma - dv_h\|_r, \\
\|d(u - u_h)\|_r &\leq C \inf_{w_h \in V^k_{h,\circ}} \|d(u - dw_h)\|_r + \eta \inf_{v_h \in V^{k-1}_{h,\circ}} \|d\sigma - dv_h\|_r + \mu \inf_{q_h \in H^k_{h,\circ}} \|p - q_h\|_r, \\
\|u - u_h\|_r &\leq C \inf_{w_h \in V^k_{h,\circ}} \|u - w_h\|_r + \eta \inf_{v_h \in V^{k-1}_{h,\circ}} \|d(u - dw_h)\|_r + \mu \inf_{q_h \in H^k_{h,\circ}} \|\sigma - v_h\|_r \\
&\quad + (\eta^2 + \delta) \inf_{v_h \in V^{k-1}_{h,\circ}} \|d\sigma - dv_h\|_r + \mu \inf_{q_h \in H^k_{h,\circ}} \|p - q_h\|_r + \mu \inf_{q_h \in H^k_{h,\circ}} \|P_{\Omega} u - w_h\|_r,
\end{align*}
\]

where $P_{\Omega} u$ denotes the $(\cdot, \cdot)_r$-orthogonal projection of $u$ onto $\Omega^k_{r,\circ}$. Furthermore, $\delta, \mu$, and $\eta$ are all of $O(h)$.

**Proof.** Since (3.9) is the domain complex of the closed Hilbert complex $(L^2, \Delta^k, d)$ satisfying the compactness property, and there exist bounded cochain projections (5.1) that are bounded uniformly as operators from $L^2(\Omega)$ to $L^2(\Omega)$ (or from $L^2(\Omega)$ to $L^k(\Omega)$) with respect to the meshsize $h$, the theorem follows by (Arnold et al., 2010, Theorem 3.11).

6. Future Work

This work can be generalized to the situation where $\Omega \subset \mathbb{R}^3$ is axisymmetric but the data (and therefore the solution) in the Hodge Laplacian problem is not axisymmetric. In that case, one can use the Fourier

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expansion with respect to the angular component \( \theta \) and solve a sequence of 2D problems instead of the original 3D problem (see Bernardi et al. (1999)). Therefore, we will have a sequence of new weighted \( L^2 \) de Rham complexes to construct and analyze.

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