**MATH235 Calculus 1**
**Quiz 2**

Show all work to receive full credit. Carefully write down your thought process. Your solution must not contain any logical errors. Good luck!

1. Find the slope of the line tangent to the curve $f(x) = x^2 - 2x - 3$ at the point $(2, -3)$ by using the limit definition of the instantaneous rate of change. Then find the equation of this tangent line.

We will calculate the limit $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ to get the general formula for the slope of the tangent line for $f(x) = x^2 - 2x - 3$. Then, we will plug in $x = 2$ to this formula to get the slope of the tangent line at the point $(2, -3)$.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{((x+h)^2 - 2(x+h) - 3) - (x^2 - 2x - 3)}{h}, \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - 2x - 2h - 3) - (x^2 - 2x - 3)}{h}, \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - 3 - x^2 + 2x + 3}{h}, \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h}, \\
&= \lim_{h\to 0} (2x + h - 2), \\
&= 2x - 2.
\end{aligned}$$

Therefore, the slop of the tangent line at the point $(2, -3)$ is $2 \times 2 - 2 = 2$.

Next, since this tangent line has slope 2, and it passes the point $(2, -3)$, by the point-slope formula, the tangent line equation is $y + 3 = 2(x - 2)$. That is, $y = 2x - 7$.

2. Find $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$ by using the Squeeze Theorem.

Since the range of sine is always between $-1$ and 1,

$$-1 \leq \sin(\frac{1}{x}) \leq 1.$$

Now, since $y = e^x$ is an increasing function, this inequality implies that

$$e^{-1} \leq e^{\sin(\frac{1}{x})} \leq e^1,$$

and by multiplying $x^2 \geq 0$ to each term above, we get

$$x^2 e^{-1} \leq x^2 e^{\sin(\frac{1}{x})} \leq x^2 e^1.$$

Hence, since

$$\lim_{x\to 0} x^2 e^{-1} = 0 = \lim_{x\to 0} x^2 e^1,$$

by the Squeeze Theorem, it follows that $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$.

3. Give an example of a function $f(x)$ that is continuous for all values of $x$ except $x = 2$, where it has a removable discontinuity. Explain how you know that $f$ is discontinuous at $x = 2$, and how you know the discontinuity is removable.

Let $f(x) = \frac{x^2 - 4}{x - 2}$. Then, $f(x) = x+2$ for all $x \neq 2$. $f$ is not continuous at $x = 2$, since the function value does not exist at $x = 2$. In particular, $f$ has a removable discontinuity at $x = 2$, since we can make $f$ continuous at $x = 2$ by defining the the function value at $x = 2$ to be 4.